Toric Fiber Products

Seth Sullivant

North Carolina State University

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Families of Ideals Parametrized by Graphs

- Let G be a finite graph
- Let R_G a polynomial ring associated to G
- Let $I_G \subseteq R_G$ an ideal associated to G

Problem

Classify the graphs G such that I_G satisfies some "nice" property.

- Often $I_G := \ker \phi_G$ for some ring homomorphism ϕ_G.

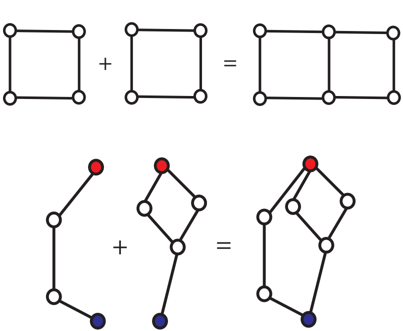
Problem

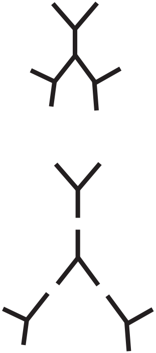
Determine generators for I_G. How do they depend on the graph G?

Graph Decompositions

Question

Are decompositions of the graphs $G = G_1 \# G_2$ reflected in the ideals $I_G = I_{G_1} \# I_{G_2}$?
How to study ideals of large graphs by breaking into simple pieces?





Toric Fiber Product

- Let $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^r\} \subset \mathbb{Z}^d$.
- Let $\mathbb{K}[x] := \mathbb{K}[x^i_j : i \in [r], j \in [s]]$, with $\deg x^i_j = \mathbf{a}^i$.
- Let $\mathbb{K}[y] := \mathbb{K}[y^i_k : i \in [r], k \in [t]]$, with $\deg y^i_k = \mathbf{a}^i$.
- Let $\mathbb{K}[z] := \mathbb{K}[z^i_{jk} : i \in [r], j \in [s], k \in [t]]$, with $\deg z^i_{jk} = \mathbf{a}^i$.
- Let $\phi : \mathbb{K}[z] \to \mathbb{K}[x] \otimes_{\mathbb{K}} \mathbb{K}[y] = \mathbb{K}[x, y]$ defined by

$$z^i_{jk} \mapsto x^i_j y^i_k \quad \text{for all } i, j, k.$$

Definition

Let $I \subseteq \mathbb{K}[x]$, $J \subseteq \mathbb{K}[y]$ ideals homogeneous w.r.t. grading by $\mathcal{A}$. The toric fiber product of I and J is the ideal

$$I \times_{\mathcal{A}} J = \phi^{-1}(I + J).$$

Two Important Examples

Example (Coarse Grading)

Let $\mathbb{K}[x]$, $\mathbb{K}[y]$ have common grading

$$\deg x_i = \deg y_j = 1 \text{ for all } i, j.$$

Then $I \subseteq \mathbb{K}[x], J \subseteq \mathbb{K}[y]$ homogeneous, are homogeneous in the standard/coarse grading.
$I \times_{\mathcal{A}} J \subseteq \mathbb{K}[z]$ is the ordinary Segre product ideal.

Example (Fine Grading)

Let $\mathbb{K}[x_1, \ldots, x_r]$, $\mathbb{K}[y_1, \ldots, y_r]$ have common grading

$$\deg x_i = \deg y_i = e_i \text{ for all } i.$$

Then $I \subseteq \mathbb{K}[x], J \subseteq \mathbb{K}[y]$ homogeneous, are monomial ideals.
$I \times_{\mathcal{A}} J = I(z) + J(z) \subseteq \mathbb{K}[z_1, \ldots, z_r]$ is the sum of monomial ideals.

A More Complex Example

Let

$$I = \langle x_{kl_1m_1}x_{kl_2m_2} - x_{kl_1m_2}x_{kl_2m_1} : k \in [r_1], l_1, l_2 \in [r_2], m_1, m_2 \in [r_3]\rangle$$

Let

$$J = \langle y_{l_1m_1n}y_{l_2m_2n} - y_{l_1m_2n}y_{l_2m_1n} :, l_1, l_2 \in [r_2], m_1, m_2 \in [r_3], n \in [r_4]\rangle$$

Let

$$\deg x_{klm} = \deg y_{lmn} = e_l \oplus e_m$$

Define $\phi : \mathbb{K}[z_{klmn} : k \in [r_1], \ldots] \to \mathbb{K}[a_{kl}, b_{km}, c_{ln}, d_{mn} : k \in [r_1], \ldots]$ by

$$z_{klmn} \mapsto a_{kl}b_{km}c_{ln}d_{mn}$$

Then $I \times_{\mathcal{A}} J = \ker \phi$.

Where's the "Fiber Product"?

- Suppose I, J are toric ideal $I = I_{\mathcal{B}}$, $J = I_{\mathcal{C}}$, $\mathcal{B} \in \mathbb{Z}^{e_1}$, $\mathcal{C} \in \mathbb{Z}^{e_2}$

$$\mathcal{B} = \{\mathbf{b}^i_j : i \in [r], j \in [s]\} \qquad \mathcal{C} = \{\mathbf{c}^i_k : i \in [r], k \in [t]\}$$

- If $I_{\mathcal{B}}$ homogeneous with respect to $\mathcal{A}$, then there is a linear map

$$\pi_1 : \mathbb{Z}^{e_1} \to \mathbb{Z}^d, \pi_1(\mathbf{b}^i_j) = \mathbf{a}^i.$$

- If $I_{\mathcal{C}}$ homogeneous with respect to $\mathcal{A}$, then there is a linear map

$$\pi_2 : \mathbb{Z}^{e_2} \to \mathbb{Z}^d, \pi_2(\mathbf{c}^i_k) = \mathbf{a}^i.$$

- Then $I \times_{\mathcal{A}} J$ is a toric ideal, whose vector configuration is a fiber product

$$\mathcal{B} \times_{\mathcal{A}} \mathcal{C}.$$

Codimension-0 Toric Fiber Products

Definition

The codimension of a TFP is the codimension of the toric ideal $I_{\mathcal{A}}$.

$I \times_{\mathcal{A}} J$ has codimension 0 iff $\mathcal{A}$ is linearly independent.

Proposition

Suppose $\mathcal{A}$ linearly independent. Let

$$m = x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} \text{ and } m' = x_{j'_1}^{i'_1} \cdots x_{j'_{n'}}^{i'_{n'}}.$$

If $\deg m = \deg m'$ *then* $n = n'$ *and*

$$i_1 = i'_1, \ldots, i_n = i'_n.$$

Persistence of Normality and Generating Sets

Theorem (Ohsugi, Michałlek, etc.)

Let $I \subseteq \mathbb{K}[x]$, and $J \subseteq \mathbb{K}[y]$ be toric ideals, and $\mathcal{A}$ linearly independent. Then

$$\mathbb{K}[z]/(I \times_{\mathcal{A}} J) \text{ normal } \Leftrightarrow \mathbb{K}[x]/I \text{ and } \mathbb{K}[y]/J \text{ normal.}$$

If $f \in \mathbb{K}[x]$, homogeneous w.r.t. $\mathcal{A}$, write

$$f = \sum c_u x_{j_{u1}}^{i_1} \cdots x_{j_{un}}^{i_n}. \qquad \text{Lift to } \sum c_u z_{j_{u1}k_1}^{i_1} \cdots z_{j_{un}k_n}^{i_n}.$$

Theorem

Let $\mathcal{A}$ linearly independent. Then $I \times_{\mathcal{A}} J$ generated by

1. *Lifts of generators of I and J*
2. *"Obvious" quadrics* $\qquad z_{j_1k_1}^i z_{j_2k_2}^i - z_{j_1k_2}^i z_{j_2k_1}^i$

GIT Quotients

- $\mathbb{N}\mathcal{A} = \{\lambda_1 \mathbf{a}^1 + \ldots + \lambda_r \mathbf{a}^r : \lambda_i \in \mathbb{N}\}$
- $R = \mathbb{K}[x]/I$ is an $\mathbb{N}\mathcal{A}$ graded ring. $R = \bigoplus_{\mathbf{a} \in \mathbb{N}\mathcal{A}} R_{\mathbf{a}}$
- $S = \mathbb{K}[y]/J$ is an $\mathbb{N}\mathcal{A}$ graded ring. $S = \bigoplus_{\mathbf{a} \in \mathbb{N}\mathcal{A}} S_{\mathbf{a}}$

Proposition

Suppose $\mathcal{A}$ linearly independent. Then

$$\mathbb{K}[z]/(I \times_{\mathcal{A}} J) = \bigoplus_{\mathbf{a} \in \mathbb{N}\mathcal{A}} R_{\mathbf{a}} \otimes_{\mathbb{K}} S_{\mathbf{a}}.$$

If $\mathbb{K} = \overline{\mathbb{K}}$, then

$$\mathrm{Spec}(\mathbb{K}[z]/(I \times_{\mathcal{A}} J)) \cong (\mathrm{Spec}(\mathbb{K}[x]/I) \times \mathrm{Spec}(\mathbb{K}[y]/J)) // T,$$

where T acts on $\mathrm{Spec}(\mathbb{K}[x]/I) \times \mathrm{Spec}(\mathbb{K}[y]/J)$ via

$$t \cdot (x, y) = (t \cdot x, t^{-1} \cdot y).$$

- Not a GIT quotient
- No hope for general construction of generators
- When is normality preserved?

Proposition

Let $\mathbb{K} = \overline{\mathbb{K}}$. Suppose that $I = \bigcap_i P_i$, and $J = \bigcap_j Q_j$ are primary decompositions. Then

$$I \times_{\mathcal{A}} J = \bigcap_{i,j} (P_i \times_{\mathcal{A}} Q_j)$$

is a primary decomposition of $I \times_{\mathcal{A}} J$.

Generators of Higher Codim TFPs

- Suppose $I \in \mathbb{K}[x]$, $J \in \mathbb{K}[y]$, are toric ideals.
- $\mathcal{A} = \{\mathbf{a}^1, \ldots, \mathbf{a}^r\}$, Let $\mathbb{K}[w] := \mathbb{K}[w_1, \ldots, w_r]$.
- Let $\psi_{xw} : \mathbb{K}[x] \to \mathbb{K}[w], \; x_j^i \mapsto w_i$ (similarly ψ_{yw})

Definition

Let

$$\tilde{I} = \langle x^{\mathbf{u}} - x^{\mathbf{v}} \in I : \phi(x^{\mathbf{u}} - x^{\mathbf{v}}) = 0 \rangle$$

$$\tilde{J} = \langle y^{\mathbf{u}} - y^{\mathbf{v}} \in J : \phi(y^{\mathbf{u}} - y^{\mathbf{v}}) = 0 \rangle$$

The ideal $\tilde{I} \times_{\tilde{\mathcal{A}}} \tilde{J}$ is the associated codimension 0 TFP.

- $\tilde{I} \times_{\tilde{\mathcal{A}}} \tilde{J} \subseteq I \times_{\mathcal{A}} J$
- $\tilde{I} \times_{\tilde{\mathcal{A}}} \tilde{J}$ is usually related (via graph theory) in a nice way to $I \times_{\mathcal{A}} J$.

Gluing Generators

Let

$$f = x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} - x_{j'_1}^{i'_1} \cdots x_{j'_n}^{i'_n} \in I$$

and

$$g = y_{k_1}^{i_1} \cdots y_{k_n}^{i_n} - y_{k'_1}^{i'_1} \cdots y_{k'_n}^{i'_n} \in J$$

that is, $\phi_{xw}(f) = \phi_{yw}(g)$. Then

$$\mathrm{glue}(f, g) = z_{j_1 k_1}^{i_1} \cdots z_{j_n k_n}^{i_n} - z_{j'_1 k'_1}^{i'_1} \cdots z_{j'_n k'_n}^{i'_n} \in I \times_{\mathcal{A}} J$$

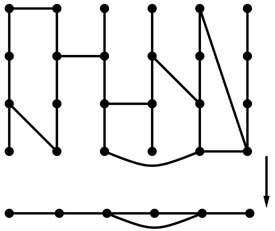
Question

Two natural classes of generators of $I \times_{\mathcal{A}} J$: when do they suffice?

Awesome Pictures that Explain Everything

Answer

Gluing and the associated codim 0 tfp always suffice to generate $I \times_{\mathcal{A}} J$. But.... how to find the right binomials to glue?



Projecting a fiber onto $\ker_{\mathbb{Z}} \mathcal{A}$.

Projected and connected fibers need not be compatible

Summary of General Results on Toric Fiber Products

- Codim 0 TFPs
  - Can Explicitly Describe Generators/ Gröbner bases from I and J
  - Normality Preserved for Toric Ideals
  - Geometric Interpretation as GIT Quotient
- Arbitrary Codim TFPs
  - Primary Decompositions “Multiply”
  - Can Explicitly Describe Generators given generators of I and J with special properties (toric case only)

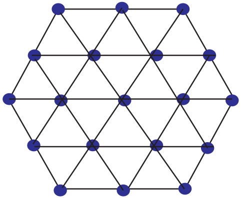
Markov Bases

Definition

Let $A : \mathbb{Z}^n \to \mathbb{Z}^d$ be a linear transformation. A Markov Basis for A is a finite subset $\mathcal{B} \subset \ker_{\mathbb{Z}}(A)$ such that for all $u, v \in \mathbb{N}^n$ with $A(u) = A(v)$ there is a sequence $b_1, \ldots, b_L \in \mathcal{B}$ such that

1. $u = v + \sum_{i=1}^{L} b_i$, and
2. $v + \sum_{i=1}^{l} b_i \geq 0$ for $l = 1, \ldots, L$.

Markov bases allow us to take random walks over the set of nonnegative integral points inside of polyhedra.



Example: 2-way tables

Let $A : \mathbb{Z}^{k_1 \times k_2} \to \mathbb{Z}^{k_1 + k_2}$ such that

$$\begin{aligned} A(u) &= \left(\sum_{j=1}^{m} u_{1j}, \ldots, \sum_{j=1}^{m} u_{k_1 j}; \sum_{i=1}^{k} u_{i1}, \ldots, \sum_{i=1}^{k} u_{ik_2} \right) \\ &= \text{vector of row and column sums of } u \end{aligned}$$

$\ker_{\mathbb{Z}}(A) = \{u \in \mathbb{Z}^{k_1 \times k_2} : \text{ row and columns sums of } u \text{ are } 0\}$
Markov basis consists of the $2\binom{k_1}{2}\binom{k_2}{2}$ moves like:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

Fundamental Theorem of Markov Bases

Definition

Let $A : \mathbb{Z}^n \to \mathbb{Z}^d$. The toric ideal I_A is the ideal

$$\langle p^u - p^v : u, v \in \mathbb{N}^n, Au = Av \rangle \subset \mathbb{K}[p_1, \ldots, p_n],$$

where $p^u = p_1^{u_1} p_2^{u_2} \cdots p_n^{u_n}$.

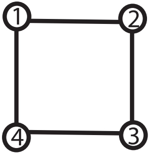
Theorem (Diaconis-Sturmfels 1998)

The set of moves $\mathcal{B} \subseteq \ker_{\mathbb{Z}} A$ is a Markov basis for A if and only if the set of binomials $\{p^{b^+} - p^{b^-} : b \in \mathcal{B}\}$ generates I_A.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow p_{21}p_{33} - p_{23}p_{31}$$

More Complicated Marginals

- Let $\Gamma = \{F_1, \ldots, F_r\}$, each $F_i \subseteq \{1, 2, \ldots, n\}$.
- Let $d = (d_1, \ldots, d_n)$ and $\mathbf{u} \in \mathbb{Z}_{\geq 0}^{d_1 \times \cdots \times d_n}$.
- Let $A_{\Gamma,d}(\mathbf{u}) = (\mathbf{u}|_{F_1}, \ldots, \mathbf{u}|_{F_r})$, lower order marginals.



2-way margins of 4-way table:
$\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}$ -margins

Question

How does the Markov basis of $A_{\Gamma,d}$ depend on Γ and d?
How do the generators of $I_{\Gamma,d}$ depend on Γ and d?

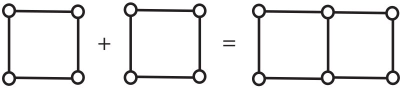
Decomposing Along Face of Simplicial Complex

Theorem (Dobra-Sullivant 2004, Hoşten-Sullivant 2002)

Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$*, and* $\Gamma_1 \cap \Gamma_2$ *is a face of both. Then*

$$I_{\Gamma,d} = I_{\Gamma_1,d_1} \times_{\mathcal{A}} I_{\Gamma_2,d_2}$$

and $\mathcal{A}$ *is linearly independent.*



These complexes are called reducible in statistics.
Allows for direct construction of Markov bases of reducible models.

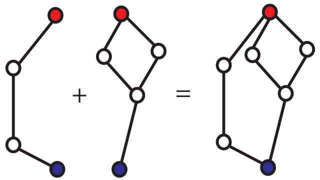
Decomposing Along Arbitrary Subcomplexes

Theorem

Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$*, and* $\Gamma_1 \cap \Gamma_2 = \Delta$*. Then*

$$I_{\Gamma,d} = I_{\Gamma_1,d_1} \times_{A_{\Delta,d'}} I_{\Gamma_2,d_2},$$

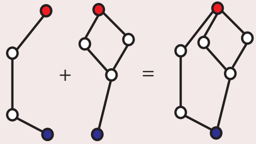
and, associated codim zero TFP is $I_{\Gamma \cup 2^{|\Delta|},d}$*.*



If overlap is "small", and associated codim zero TFP is "simple", can construct generators of $I_{\Gamma,d}$.

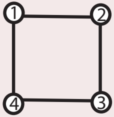
Example Applications

Theorem (Kral, Norin, Pragnac 2010)



If $d_i = 2$ for all i, then $I_{\Gamma,d}$ is generated in degree ≤ 4 if Γ is a series-parallel graph (no K_4-minors).

Theorem (4-cycle)



If $d_2 = d_4 = 2$, then $I_{\Gamma,d}$ generated in degree 2 and 4, for all d_1, d_3.

Theorem

If Γ is the boundary of a bipyramid over a simplex of dimension d, and all $d_i = 2$, $I_{\Gamma,d}$ generated in degrees $2^{d+1}, 2^d, 2$.

Conditional Independence Ideals

- $X = (X_1, \ldots, X_n)$ is an n-dimensional discrete random vector.
- Probability Distribution

$$p_{i_1 \cdots i_n} = \mathrm{Prob}(X_1 = i_1, \ldots, X_n = i_n)$$

- Let (A, B, C), partition of $[n]$.
- Conditional independence statement $X_A \perp\!\!\!\perp X_B | X_C$ gives a binomial ideal in $\mathbb{C}[p_{i_1 \cdots i_n} : i_j \in [r_j]]$.

Example

Let $n = 4$, $r_1 = r_2 = r_3 = r_4 = 2$, $X_1 \perp\!\!\!\perp X_3 | (X_2, X_4)$, gives CI ideal

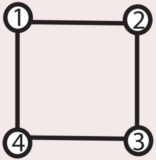
$$\begin{aligned} I_{X_1 \perp\!\!\!\perp X_3 | (X_2, X_4)} &= \langle p_{1111}p_{2121} - p_{1121}p_{2111}, p_{1112}p_{2122} - p_{1122}p_{2112}, \\ &\quad p_{1211}p_{2221} - p_{1221}p_{2211}, p_{1212}p_{2222} - p_{1222}p_{2212} \rangle \end{aligned}$$

Global Conditional Independence Ideals

- G undirected graph, vertex set $[n]$
- CI statement $X_A \perp\!\!\!\perp X_B | X_C$ holds for G if C separates A from B in G.
- Let $\mathrm{Global}(G)$ set of all CI statements holding for G.

$$I_{\mathrm{Global}(G)} = \sum_{X_A \perp\!\!\!\perp X_B | X_C \in \mathrm{Global}(G)} I_{X_A \perp\!\!\!\perp X_B | X_C}$$

Example (4-cycle)



$$I_{\mathrm{Global(G)}} =$$
$$I_{X_1 \perp\!\!\!\perp X_3 | (X_2, X_4)} + I_{X_2 \perp\!\!\!\perp X_4 | (X_1, X_3)}$$

Clique Sums

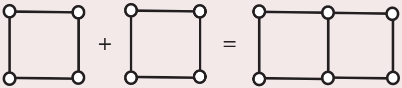
Proposition

If $G = G_1 \# G_2$ is a clique sum, then

$$I_{\mathrm{Global(G)}} = I_{\mathrm{Global(G_1)}} \times_{\mathcal{A}} I_{\mathrm{Global(G_2)}}$$

and $\mathcal{A}$ is linearly independent.

Example



For $r_i = 2$ for all i, $I_{\mathrm{Global}(G)}$ has $9 \times 9 = 81$ prime components.

Conjecture

For all graphs G, $I_{\mathrm{Global}(G)}$ is a radical ideal.
All nontoric components are related to marginal positivity.

Summary

- Problems in Algebraic Statistics call for the study of ideals associated to graphs and simplicial complexes
- Graph theory provides decomposition theory
- Algebraic structure of ideals reflected in structure of graphs (conjecturally)
- Toric fiber product is an algebraic decomposition tool for proving results for some graph classes

References

- A. Engström and T. Kahle. Multigraded commutative algebra of graph decompositions. `1102.2601`
- S. Sullivant. Toric fiber products. *J. Algebra* **316** (2007), no. 2, 560–577. `math.AC/0602052`