Math 526 – Algebraic Geometry – Learning Objectives

This handout describes specific learning objectives for Math 526, Algebraic Geometry. What is a learning objective? This is something that you should *know* and also something you should be able to *do* by the end of the class. In particular, these are the things that you must be able to do for the exams.

1. Material up to the Midterm

By the time of the midterm, a student in Math 526 will be able to:

- *State* the definition of a monomial order. *Verify* whether certain orderings on monomials are monomial orders. *Compute* leading terms of given polynomials with respect to given monomial orders.
- *Perform* polynomial long division.
- *State* definition of Gröbner basis.
- *State* Buchberger's criterion and Buchberger's algorithm. *Use* these to check whether a given set of polynomials is a Gröbner basis or not.
- *State* definition of reduced Gröbner basis and *check* whether a given
- *State* definition of a Noetherian ring. *State and prove* Noether's proposition (equivalence between finite generation of ideals and ascending chain condition).
- *State* the Hilbert basis theorem, and outline its proof. *Use* Hilbert basis theorem to prove existence of Gröbner bases.
- *Use* computer algebra system to compute Gröbner bases.
- *State* main definitions of algebraic geometry: vanishing ideal and affine variety. *Compute* vanishing ideal and draw pictures of varieties in small examples.
- *Prove* basic properties of vanishing ideals and varieties under various operations: intersection and union of varieties, intersection and sum of ideals.
- *Define* Zariski closure of a set in affine space and compute it in small examples.
- *Define* and *construct* interesting morphisms of varieties.
- *Define* the coordinate ring of a variety. *Relate* morphisms of varieties to maps between coordinate rings.
- *Define* and *construct* interesting rational maps of varieties.
- *Define* the ring of fractions of a variety. *Relate* rational maps of varieties to maps between rings of fractions.
- *State* and *prove* main results on elimination theory: Elimination theorem, relation to coordinate projections.
- *Define* the graph of a morphism and a rational map. *Explain* how to set up a Gröbner basis computation to compute the vanishing ideal of a morphism or rational map.
- *Use* computer algebra systems to compute various operations in algebraic geometry: Gröbner bases, images of morphisms and rational map, vanishing ideal of parametrizations.
- *Define* and work with familiar examples of algebraic varieties: secant varieties, joins, cone over a point, product varieties, birational varieties.
- *State* three versions of Nullstellensatz and *prove* equivalance between these notions.

2. Material up to the Midterm

By the end of the semester, a student in Math 526 will be able to:

- *State* and *apply* main results on irreducible decomposition of varieties, and its connection to Noetherian rings.
- *Decompose* a variety into its irreducible pieces in small examples.
- *State* and *apply* main results on primary decomposition of ideals.
- *Define* important parts of the primary decomposition: associated prime, minimal prime, embedded prime, primary component, radical of an ideal, ideal quotient (colon ideal).
- *Use* primary decomposition to identify zero divisors in a ring.
- *Compute* primary decompositions in small examples.
- *Explain* how algebraic notion of primary decomposition relates to geometric notion of irreducible decomposition of varieties.
- *Compute* primary decompositions of monomial ideals. *Characterize* a given monomial ideal as radical, irreducible, primary, prime, or none of these.
- *Define* field concepts related to dimensions of varieties (e.g. algebraic independence, transcendence basis, etc.). *Calculate* the dimension of varieties in small example.
- *Define* projective space and its distinguished affine open sets (charts/patches).
- *Define* homogeneous polynomials, homogeneous ideals, homogenization, dehomogenization.
- *Compute* homogenization/ dehomogenizations of individual elements and of ideals (using Gröbner bases in the latter case).
- *Define* projective varieties in two ways, and explain why these definitions are equivalent.
- *Define* homogeneous vanishing ideal of a projective variety, and Zariski closure in projective space. *Compute* the Zariski closure in small examples, and explains how it related to homogenization in general.
- *Define* irreducible variety, relate to prime ideals.
- *State* two versions of the projective Nullstellensatz and define related concepts (e.g. irrelevant ideals). *Sketch* its proof by by reducing to the affine case.
- *Define* morphisms and rational maps of projective varieties.
- *Determine* whether or not a given rational map is a morphism or not.
- *Compute* the homogeneous vanishing ideal of the image of a morphism or rational map using Gröbner bases.
- *Define* and work with familiar examples of projective varieties defined via morphisms, (i.e. rational normal curve, Veronese variety, Segre variety, Grassmannian). *Sketch* the construction of equations of these varieties, especially in small examples.
- *Use* linear algebra arguments to describe the homogeneous vanishing ideals of varieties related to Grassmannians (e.g. set of all pairs of linear spaces with nontrivial intersection, set of planes containing a line, etc.).

- *Define* graded rings, modules, the Hilbert function, and Hilbert polynomials. *Identify* familiar examples of graded rings and modules (rings, ideals, quotient rings, direct sums, the d-th twist $R(-d)$, homogeneous coordinate ring of a variety).
- *Use* properties of Gröbner bases to reduce Hilbert function calculation to the case of monomials ideals. *Compute* Hilbert functions/polynomials of monomial ideals in small examples.
- *Define* exact sequences of modules. *Use* exact sequences of modules to calculate Hilbert functions, (e.g. the exact sequence for a nonzero divisor, the exact sequence for ideal sum).
- *Define* dimension and degree of a projective variety, as they are related to the Hilbert polynomial. *Compute* dimension and degree of varieties in specific examples (e.g. hypersurfaces, Segre varieties, product varieties $V \times W$, complete intersections).
- *Define* the notion of a generic property on a variety.
- *Define* the multiplicity of a point (that is, of an ideal primary to a maximal ideal).
- *Define* the degree of a projective variety, as it is related to intersecting with generic planes. *Sketch* the idea of the proof of the equivalence of the two notions of degree (i.e. using the exact sequence for a nonzero divisor).
- *State* Bezout's theorem and apply it in small examples.