Math 521 – Learning Objectives

This handout describes specific learning objectives for Math 521 and (eventually) for Math 721. What is a learning objective? This is something that you should *know* and also something you should be able to *do* by the end of the class. In particular, these are the things that you must be able to do for the exams, including the qualifying exams.

1. Material up to the first midterm

By the time of the first midterm, a student in Math 521 will be able to:

- *State* the definition of a group. Use the definition to *prove* that a given (possibly abstract) set with a binary operation/composition law is a group or is not a group. *Show* that a given group is abelian or is not abelian. *Compute* the multiplication table of a group.
- *State* the definition of a subgroup. Use the definition to *prove* that a given (possibly abstract) subset of a group is a subgroup or is not a subgroup.
- *State* the definition of a subgroup $\langle U \rangle$ generated by U. *Calculate* the subgroup generated by $U \subseteq G$ in specific examples.
- *State* the definition of group homomorphism, isomorphism, automorphism, endomorphism. Use the definition to *prove* or disprove that a given map is a homomorphism, isomorphism, automorphism, or endomorphism (depending on the example). *Compute with* homomorphisms including computing the kernel or image of a homomorphism.
- Given two groups G and H *show* that they are isomorphic or not isomorphic.
- *Compute* the order of a group, the order of an element of a group, and the order of a subgroup of a group in specific examples.
- *Compute* all automorphisms of a given group G and *classify* the automorphism group $\mathrm{Aut}(G)$ up to isomorphism.
- *State* the definition of a normal subgroup. Use the definition to *prove* that a given (possibly abstract) subgroup of a group is a normal subgroup or not.
- *Construct* and *compute with* direct products.
- *State* the fundamental theorem of finitely generated abelian groups and use it to *classify* finite abelian groups of a given order up to isomorphism.
- In specific examples, *detect* if a group has the form $G \cong H \times K$.
- Use the mapping property of products to *analyze* homomorphisms into a direct product.
- *Construct* and *compute with* semidirect products.
- For a given pair of groups H, K *classify* all possible semidirect products $H \rtimes K$ up to isomorphism.
- In specific examples, *detect* if a group has the form $G \cong H \rtimes K$.
- *State* the definition of equivalence relation and partition and use the definition to *prove* that a given relation is an equivalence relation.
- *State* the definition of a left or right coset. *Construct* all the left and right cosets of a subgroup H of a group G. *Prove* properties of left and right cosets.
- *State* Lagrange's theorem and use it to *prove* properties about subgroups and homomorphisms.

- Use the coset characterization of normality to *detect* if a subgroup is normal.
- For a normal subgroup $H \trianglelefteq G$, *compute with* the quotient group G/H and *identify* the group G/H up to isomorphism.
- *State* the first isomorphism theorem and use it to identify a quotient group up to isomorphism.
- *Compute with* familiar examples of groups including: the cyclic groups C_n and $\mathbb{Z}$; the dihedral group D_n; the symmetric group S_n; the additive and multiplicative groups of fields; matrix groups including: the general linear group $Gl_n(\mathbb{K})$, special linear group $Sl_n(\mathbb{K})$, the Borel group of upper triangular matrices $\mathcal{B}$.
- *Compute with* integers modulo n (that is computations in $\mathbb{Z}/n\mathbb{Z}$).
- *Compute with* the symmetric group S_n. *Convert* between 2-line notation and cycle notation. *Multiply* elements of symmetric group in both 2-line notation and cycle notation. Given a set of elements $U \subseteq S_n$ *determine* the subgroup generated by U.
- *State* and *apply* the classification of conjugacy classes in the symmetric group.
- *Compute with* the alternating group A_n.
- *Construct* examples of groups, subgroups, homomorphisms, etc. that satisfy or violate specific properties that a group, subgroup, homomorphism etc. might have.

2. Material Between First and Second Midterms

By the time of the second midterm, a student in Math 521 will be able to do everything above and:

- *State* the definition of a group action. In particular examples, *show* whether or not a (possibly abstract) example of a composition law is a group action.
- *Use* and *compute with* familiar group actions including: the action of a group on itself by left multiplication, the action of a group on itself by conjugation, the action of the dihedral group on subsets of the n-gon, the action of S_n on $[n]$, the groups of rotational symmetries of the cube and the icosahedron.
- *State* the definition of an orbit and the stabilizer of a group action. In particular examples, *compute* the orbit and stabilizer of an element.
- *State* and *prove* the Orbit-Stabilizer Theorem.
- *Use* the orbit-stabilizer theorem to count the number of elements in a symmetry group.
- *Deduce* consequences of the orbit-stabilizer theorem for familiar group actions including information about conjugacy classes.
- *Compute* the class of equation of given groups. Use properties of the class equation to deduce properties of groups of small order.
- *Use* group actions and the class equation to prove properties of p-groups.
- *State* the definition of the center of a group. *Show* that the center of a group is nontrivial in particular (possibly abstract) examples. *Use* the existence of a nontrivial center of a group to deduce properties of the group.
- *State* the Sylow theorems.
- *Compute* the Sylow p-subgroups of a group G in specific examples.
- *Use* the Sylow theorems to show that groups of certain orders cannot be simple.

- *Use* the Sylow theorems and semidirect products to classify all groups of a certain order.
- *Compute with* the free group.
- *Determine* the isomorphism class or basic properties of groups given by generators and relations.
- *State* the definition of a ring. Use the definition to *prove* that a given (possibly abstract) possibly abstract set with two composition laws is a ring or not. *Show* that a ring is commutative or not.
- *State* the definition of a subring. *Verify* that a given (possibly abstract) subset of a ring is a subring or not.
- *State* the definition of a field. Use the definition to *prove* that a given (possibly abstract) possibly abstract set with two composition laws is a field or not.
- *Compute* with familiar examples of rings including the integers $\mathbb{Z}$, fields, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_p$, integers modulo n $\mathbb{Z}/n\mathbb{Z}$, matrix rings $M_{n\times n}(R)$, polynomial rings $R[x]$, and $R[\mathbf{x}] = R[x_1, \ldots, x_n]$, the ring of formal power series $R[[x]]$, and subrings of these objects.
- *State* the definition of a ring homomorphism, isomorphism, endomorphism, or automorphism. *Prove* that a given map between rings is a ring homomorphism, isomorphism, endomorphism, or automorphism or not.
- In specific examples, *prove* that two rings are isomorphic or not.
- *State* and *use* the definition of the evaluation homomorphism from a polynomial ring.
- *State* the definition of ideal. In specific examples, *prove* that a given subset of a ring is an ideal (or not). *Explain* why ideals are a useful notion in ring theory.
- *Compute* the kernel of a ring homomorphism.
- *Compute* the characteristic of a given ring.
- For a given ring R and ideal I, *state* the definition of the quotient ring R/I. *Compute* in the quotient ring R/I. *Identify* the isomorphism type of the quotient ring in specific examples if possible.
- *Apply* the division algorithm to prove statements about ideals.
- *State* the first and third isomorphism theorems for quotient rings and the correspondence theorem for ideals in quotient rings, and *use* these theorems to determine the structure of a quotient ring.
- *Construct* examples of rings, subrings, homomorphisms, ideals etc. that satisfy or violate specific properties that a ring, subring, homomorphism, ideal etc. might have.

3. Material Between Second Midterm and Final

- *State* the definition of important classes of ring elements including: unit, nilpotent, unipotent, zero divisor, associates, irreducible, prime. *Classify* such elements in a given ring. Use the properties of such important classes.
- *State* and *identify* important classes of ideals including: prime ideal, radical ideal, maximal ideal, primary ideal, irreducible ideal. *Relate* these classes of ideals to

teach other and in particular examples, *identify* an ideal as falling into one of these classes.

- *Compute* in rings that are obtained by adjoining elements.
- *State* and *identify* important classes of rings including: integral domain, local ring.
- *Compute* the field of fractions of a given integral domain. *Use* the universal property of the field of fractions.
- *State* and *use* the basic properties of factorization of elements. *Relate* the ascending chain condition for principal ideals to factorization.
- *Relate* element types associated with factorization to properties of ideals.
- *State* definitions of important classes of rings associated with factorization: Unique Factorization Domains, Principal Ideal Domains, Noetherian Rings. *Show* that a given (possibly abstract) ring falls into one of these classes (or not).
- *Use* Gauss's Lemma and its corollaries to factorize polynomials in $\mathbb{Z}[x]$ and related rings.
- *Use* ring homomorphisms and Eisenstein's criterion to analyze factorizations of polynomials. *Prove* that prime cyclotomic polynomials are irreducible.
- *State* and *use* equivalent characterizations of Noetherian rings.
- *State* Hilbert's Basis Theorem and use it to show that specific rings are Noetherian. *Outline* the proof of Hilbert's Basis theorem.
- In small examples, *use* Hilbert's Basis Theorem and its proof to construct generating sets of ideals (especially the kernel of a ring homomorphism).
- *State* the definition of primary decomposition. In simple examples, give a decomposition into irreducible and primary ideals.
- *State* the definition of colon ideal (quotient ideal), primary ideal, associated prime, and radical of an ideal. *Compute* these operations in simple examples.
- *Construct* examples of rings, ideals, and homomorphisms etc. that satisfy or fail to satisfy the above properties that a ring, ideal, or homomorphism might have.
- *Compare* properties that rings, ideals, homomorphisms etc. might have. (e.g. Prime ideals are irreducible, is every irreducible ideal a prime ideal?)