Differential Independence of Solutions of a Class of q-Hypergeometric Difference Equations

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This note is a companion to the paper "Differential Galois Theory of Difference Equations" by the same authors. In a previous version of this note (and in Example 3.14 of this paper) we claimed that for the q-hypergeometric equation with parameters restricted as below, the associated \( \sigma\delta \)-PV extension has differential transcendence degree 3. We have since found gaps in the proof of this claim. Nonetheless, we are able to show this claim when we further restrict the parameters. In the following we show how MAPLE can be used to verify these cases.

We shall consider the equations

\[
y(q^2x) - [(a+b)x-(1+c/q)]/(abx -c/q)]y(qx) + [(x-1)/(abx -c/q)]y(x) = 0
\]

where \( a = b, \ c=q, \ a \notin q \mathbb{Z}, \ a^2 \in q \mathbb{Z} \). We shall show in the cases when \( a^2 = q, q^3, q^5 \) that the associated \( -\)PV extension has differential transcendence degree 3. The MAPLE code we give below can be easily altered to test for other values of \( a^2 \). We suspect that this result is true in general but, as we said above, we do not have a proof.

In the paper "Differential Galois Theory of Difference Equations", we showed that for a difference equation \( \sigma(Y) = AY \) over \( \mathbb{C}(x) \) with difference Galois group \( \text{SL}_n(\mathbb{C}) \), the differential transcendence degree is less than \( n^2 - 1 \) if and only if there exists a matrix \( B \) in \( \text{gl}_n(\mathbb{C}(x)) \) such that \( \sigma(B) = ABA^{-1} + \delta A A^{-1} \), where \( \delta = x/dx \). This latter equation is an \( n^2 \times n^2 \) system in the entries of \( B \). In section 1, we shall derive this system for the above family of q-hypergeometric equations. In section 2, we shall derive a scalar equation for one of the entries of \( B \).

1. The equation for \( B \).

The matrix equation associated with our scalar equation above is \( \sigma(Y) = AY \). We calculate A below.

\[
> \text{with(LinearAlgebra)}:
\]
We now calculate the system associated to \( \sigma(B) = ABA^{-1} + \delta A A^{-1} \).
The associated system for B (now written as a column vector as above) is $\sigma(B) = CB+E$, where $C$ and $E$ are as above.

2. The scalar equation for $v$. 
To compute a scalar equation for the component $v$ of the vector $B$, we proceed as follows. We will compute matrices $C_i$ and $E_i$ such that $\sigma_i(B) = C_iB + E_i$. Let $c_i$ denote the entry in second row of $C_i$ and $e_i$ denote the entry of the second row of $E_i$. We will find elements $y_i$ such that $\Sigma y_i c_i = 0$. We then will have the scalar equation $\Sigma y_i \sigma_i(v) = \Sigma y_i e_i$. Note that if $e = (0,1,0,0)$, then for any matrix $M$, $eM$ is the second row of $M$.

```plaintext
> e:=Matrix([[0,1,0,0]]);
e := [ 0 1 0 0 ]

> s1C:=subs(x=q*x,C);
s1C :=
   0 0  \frac{2 (x q a - 1)}{-1 + x q} 1
   0 0  - \frac{a^2 x q - 1}{-1 + x q} 0
   \frac{-2 (x q a - 1)}{a^2 x q - 1} - \frac{-1 + x q}{a^2 x q - 1}
   1 0  - \frac{2 (x q a - 1)}{-1 + x q} 0

> s2C:=subs(x=q*x,s1C);
s2C :=
   0 0  \frac{2 (x q^2 a - 1)}{-1 + x q^2} 1
   0 0  - \frac{a^2 x q^2 - 1}{-1 + x q^2} 0
   \frac{-2 (x q^2 a - 1)}{a^2 x q^2 - 1} - \frac{-1 + x q^2}{a^2 x q^2 - 1}
   1 0  - \frac{2 (x q^2 a - 1)}{-1 + x q^2} 0

> s3C:=subs(x=q*x,s2C);
s3C :=
```

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> (13)

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> (14)

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> (15)

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> (16)

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Since $C_0 = I$, we have $c0:=e$.

Since $C_1 = C$, we have $c1:=\text{Multiply}(e,C)$.

Since $C_2 = \sigma(C)C$, we have $c2:=\text{simplify}(\text{Multiply}(e,\text{Multiply}(s1C,C)))$.

Since $C_2 = \sigma^2(C)\sigma(C)C$, we have $c3:=\text{simplify}(\text{Multiply}(e,\text{Multiply}(s2C,\text{Multiply}(s1C,C))))$.

A priori, there will be a dependence among $e0,e1,e2,e3,e4$ but in this case there is actually a dependence among $e0,e1,e2,e3$. To find such a dependence let $MM:=\text{Transpose}((\begin{array}{c} c0 \\ c1 \\ c2 \\ c3 \end{array}))$.

The dependence will be given by the following element of the nullspace of this matrix.

$\begin{align*}
\text{VV} &:= (\text{Transpose}(\text{op}(1,\text{NullSpace}(MM))))
\end{align*}$
\[
V\nu := \begin{bmatrix}
- \frac{(a^2 + x^2 q^2 - 1) (x q a - 1)}{(a^2 x q - 1) (-1 + x q^2)}, \\
\frac{(a^2 x + 3 + 3 x^2 q a^2 + x q - 4 x q a - 4 x a) (a^2 x q^2 - 1)}{(a^2 x - 1) (a^2 x q - 1) (-1 + x q^2)}, \\
\frac{(a^2 x q^2 - 1) (x q a - 1) (a^2 x + 3 + 3 x^2 q a^2 + x q - 4 x q a - 4 x a)}{(a^2 x - 1) (a^2 x q - 1)^2 (x a - 1)}, \\
\end{bmatrix}
\]

Letting \(y_{i-1}\) be the \(i\)th entry of this vector, we have that \(\sigma^3(v) + y_2 \sigma^2(v) + y_1 \sigma(v) + y_0 v = e_3 + y_2 e_2 + y_1 e_1 + y_0 e_0\). Clearly \(e_0 = e_1 = 0\). A calculation shows that \(E_2 = \sigma(C)E + \sigma(E)\) and \(E_2 = \sigma^2(C)\sigma(C)E + \sigma^2(C)\sigma(E) + \sigma^2(E)\) so

\[
> e0 := 0;
\]

\[
> e1 := (e.E)[1,1];
\]

\[
> e2 := (\text{Multiply}(e, \text{Multiply}(s1C, E) + s1E))[1,1];
\]

\[
e2 := \frac{2 (a^2 x q - 1) (-1 + a) x}{(1 + x q) (a^2 x - 1) (-1 + x)}
\]

\[
> e3 := \text{Multiply}(e, (\text{Multiply}(s2C, \text{Multiply}(s1C, E)) + \text{Multiply}(s2C, s1E) + s2E))[1,1];
\]

\[
e3 := -\frac{1}{-1 + x q^2} \left(\frac{(a^2 x q^2 - 1)}{(a^2 x q - 1)} \left(\frac{8 (x q a - 1)^2 (-1 + a) x}{(a^2 x q - 1) (-1 + x q) (a^2 x - 1) (-1 + x)} + \frac{2 (x q a - 1) x (-1 + a^2)}{(a^2 x q - 1) (a^2 x - 1) (-1 + x)}\right) + \frac{2 (a^2 x q^2 - 1) (-1 + a) x q}{(1 + x q^2) (a^2 x q - 1) (-1 + x q)}\right)
\]

\[
> EE := \text{Matrix}([[e0], [e1], [e2], [e3]]);
\]

\[
EE := \begin{bmatrix}
0, \\
0, \\
0, \\
\frac{2 (a^2 x q - 1) (-1 + a) x}{(1 + x q) (a^2 x - 1) (-1 + x)}, \\
-\frac{1}{-1 + x q^2} \left(\frac{(a^2 x q^2 - 1)}{(a^2 x q - 1)} \left(\frac{8 (x q a - 1)^2 (-1 + a) x}{(a^2 x q - 1) (-1 + x q) (a^2 x - 1) (-1 + x)} + \frac{2 (x q a - 1) x (-1 + a^2)}{(a^2 x q - 1) (a^2 x - 1) (-1 + x)}\right) + \frac{2 (a^2 x q^2 - 1) (-1 + a) x q}{(1 + x q^2) (a^2 x q - 1) (-1 + x q)}\right)
\end{bmatrix}
\]
We wish to show that this equation has no nonzero rational solutions. In order to use MAPLE taking into account that $a^2$ is a power of $q$. To do this we will replace $q$ by $q^2$ and consider a sixth order equation.

\[ \sigma^3(v) + y_2\sigma^2(v) + y_1\sigma(v) + y_0v = e_3 + y_2 e_2 + y_1 e_1 + y_0 e_0 \]
equation. When the output is empty, we can conclude that there are no nonzero solutions. The MAPLE command RationalSolution will yield a basis of the rational solutions of a q-difference equation. This avoids having to deal with the square root of \(q\) when \(a^2\) is an odd power of \(q\). We then apply the MAPLE command RationalSolution. This command will return an empty set when there are no rational solutions.

```maple
> EQN0:=subs(q=t,EQN):
EQN0:=-(-1 + x t) \((a^2 x t - 1) (a^2 x - 1) (-1 + x) (a^2 x t^2 - 1) (x t a - 1) v(x) + (-1 + x t) \((a^2 x t - 1) (x a - 1) (a^2 x + 3 x^2 t a^2 + x t - 4 x t a)
- 4 x a) (a^2 x t^2 - 1) v(x t) - (-1 + x t) \((a^2 x - 1) (a^2 x t^2 - 1) (x t a - 1) (a^2 x + 3 + 3 x^2 t a^2 + x t - 4 x t a - 4 x a) v(x t^2) + (-1 + x t) \((a^2 x t - 1)^2 (a^2 x t^2 - 1) v(x t^3) = 2 \((a^2 x t - 1) (x^2 t a^2 + x^2 t a^4 - x^2 t^2 a^3
- x^2 t a^3 + a - a^2 - t + t a) (a^2 x t^2 - 1) x
> EQN1:=subs(a=q^1,t=q^2,EQN0);
EQN1:=-(-1 + x q^2)^2 \((q^4 x - 1) (-1 + x) (q^6 x - 1) (-1 + x q^3) v(x) + (-1 + x q^2) \((q^4 x - 1) (-1 + x q) \((2 x q^2 + 3 + 3 x^2 q^4 - 4 x q^3 - 4 x q) (q^6 x
- 1) v(x q^2) - (-1 + x q^2)^2 \((q^4 x - 1) (-1 + x q^3) \((2 x q^2 + 3 + 3 x^2 q^4 - 4 x q^3
- 4 x q) v(x q^4 x) + (-1 + x q^2)^2 \((q^4 x - 1)^3 (-1 + x q) v(q^6 x) = 2 \((q^4 x
- 1) \((2 x^2 q^6 - x^2 q^7 - x^2 q^5 + q - 2 q^2 + q^3) (q^6 x - 1) x
> with(QDifferenceEquations):
The MAPLE command RationalSolution will yield a basis of the rational solutions of a q-difference equation. When the output is empty, we can conclude that there are no nonzero solutions.
> sol1 := RationalSolution(EQN1, v(x), {}, output = basis[_K]);
sol1 :=
> EQN3:=subs(a=q^3,t=q^2,EQN0):
with(QDifferenceEquations):
sol1 := RationalSolution(EQN3, v(x), {}, output = basis[_K]);
sol1 :=
> EQN5:=subs(a=q^5,t=q^2,EQN0):
with(QDifferenceEquations):
sol1 := RationalSolution(EQN5, v(x), {}, output = basis[_K]);
sol1 :=
> EQNminus1:=subs(a=q^(-1),t=q^2,EQN0):
with(QDifferenceEquations):
sol1 := RationalSolution(EQNminus1, v(x), {}, output = basis[_K])
;
sol1 :=
> EQNminus3:=subs(a=q^(-3),t=q^2,EQN0):
with(QDifferenceEquations):
sol1 := RationalSolution(EQNminus3, v(x), {}, output = basis[_K])
;
sol1 :=
> EQNminus5:=subs(a=q^(-5),t=q^2,EQN0):
with(QDifferenceEquations):
sol1 := RationalSolution(EQNminus5, v(x), {}, output = basis[_K])
;
\[ \text{sol1} := \]

\[ > \text{EQN2} := \text{subs}(a = -q, t = q, \text{EQN0}); \]
\[ \text{with(QDifferenceEquations);} \]
\[ \text{sol1} := \text{RationalSolution(EQN2, v(x), \{}\}, \text{output} = \text{basis[_K]}); \]
\[ \text{sol1} := \]