Mahler equations and rationality

Reinhard Schäfke,* Michael F. Singer‡

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Abstract

We give another proof of a result of Adamczewski and Bell [1] concerning Mahler equations: A formal power series satisfying a p- and a q-Mahler equation over $\mathbb{C}(x)$ with multiplicatively independent positive integers p and q is a rational function. The proof presented here is self-contained and essentially a compilation of proofs contained in a recent preprint [9] of the authors.

Keywords. Linear difference equations, consistent systems, q-difference equation, Mahler equation

We consider two Mahler operators, *i.e.* two endomorphisms σ_j, $j = 1, 2$, on the field $K = \mathbb{C}[[x]][x^{-1}]$ of formal Laurent series with complex coefficients defined by $\sigma_1(f(x)) = f(x^p)$, $\sigma_2(f(x)) = f(x^q)$ for any $f(x) \in K$ where p and q are positive integers. Observe that σ_1 and σ_2 commute, *i.e.* $\sigma_1\sigma_2 = \sigma_2\sigma_1$. We consider the field $\mathbb{C}(x)$ of rational functions with complex coefficients as a subfield of K, the inclusion given by the expansion in a Laurent series at the origin. We want to prove the following theorem

Theorem 1. *Assume that p and q are multiplicatively independent, i.e. there are no nonzero integers n_j such that $q^{n_2} = p^{n_1}$. Suppose that the formal series $f(x) \in K$ satisfies a system of two Mahler equations*

$$S_j(f(x)) = \sigma_j^{m_j}(f(x)) + b_{j,m_j-1}(x)\sigma_j^{m_j-1}(f(x)) + \ldots + b_{j,0}(x)f(x) = 0, \ j = 1, 2 \qquad (1)$$

with $b_{j,i}(x) \in \mathbb{C}(x)$.

Then $f(x)$ is rational.

Remark: 1. This Theorem was recently proved by Adamczewski and Bell in [1]. Their tools include a local-global principle to reduce the problem to a similar problem over finite fields, Chebotarev's Density Theorem, Cobham's Theorem and some asymptotics - all very different from the techniques used in the present work.
2. [1] also provides background information about Mahler equations, in particular historical, many

*Institut de Recherche Mathématique Avancée, Université de Strasbourg et C.N.R.S., 7, rue René Descartes, 67084 Strasbourg Cedex, France, schaefke@unistra.fr.

‡Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695, USA, singer@ncsu.edu.

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references to the literature and explains the relation to Cobham's theorem in the theory of finite state machines. The fact that the generating functions of p-regular (and thus of p-automatic sequences) satisfy p-Mahler equations is shown in [3].
3. The subsequent proof is essentially a compilation of work contained in our recent preprint [9], see Corollary 15, part 3, and Proposition 19. The preprint presents a unified reduction theory of consistent pairs of first order systems of linear differential, difference, q-difference or Mahler equations like the one of Proposition 3 below and uses it to deduce numerous statements on common solutions of two scalar linear differential, difference, q-difference or Mahler equations.

Proof. We may assume without loss of generality that $b_{j,0}(x) \neq 0$, $j = 1, 2$. This follows from

Lemma 2. *Consider* $w_1(x), ..., w_\ell(x) \in K$ *and a positive integer* m. *Then these series are* $\mathbb{C}(x)$-*linearly dependent if and only if* $w_1(x^m), ..., w_\ell(x^m)$ *are.*

Proof. We only prove the nontrivial implication. Suppose that $w_1(x^m), ..., w_\ell(x^m)$ are $\mathbb{C}(x)$-linearly dependent, which means that there exist $a_k \in \mathbb{C}(x)$, $k = 1, ..., \ell$, not all zero, such that

$$a_1(x)w_1(x^m) + ... + a_\ell(x)w_\ell(x^m) = 0.$$

Now we can uniquely write $a_k(x) = \sum_{j=0}^{m-1} x^j c_k^j(x^m)$, $k = 1, ..., \ell$, with rational functions $c_k^j(x)$. Expanding the terms in the above equation in Laurent series we obtain the equations

$$c_1^j(x^m)w_1(x^m) + ... + c_\ell^j(x^m)w_\ell(x^m) = 0, j = 0, ..., m-1,$$

and hence

$$c_1^j(x)w_1(x) + ... + c_\ell^j(x)w_\ell(x) = 0, j = 0, ..., m-1.$$

Since at least one of them must be nontrivial we obtain the linear dependence of the w_j, $j = 1, ..., \ell$. ∎

Consider now the $\mathbb{C}(x)$-subspace W of K generated by $\sigma_1^m \sigma_2^r(f)$, $m = 0, ..., m_1 - 1$, $r = 0, ...m_2 - 1$. By (1), W is invariant under σ_1 and σ_2; here the fact that the σ_j commute is used.

Let $g_1, ..., g_n$ be a $\mathbb{C}(x)$-basis of W with $g_1 = f$ and let $g = (g_1, ..., g_n)^T$. Then we have that

$$\sigma_1(g) = A(x)g,\ \sigma_2(g) = B(x)g, \tag{2}$$

with $A, B \in \mathrm{gl}_n(\mathbb{C}(x))$. By Lemma 2, we actually have $A, B \in \mathrm{GL}_n(\mathbb{C}(x))$ because the components of $\sigma_j(g)$ form again a basis of W.

Additionally, the coefficient matrices of (2) satisfy a certain *consistency condition*. Indeed, we have

$$0 = \sigma_1(\sigma_2(g)) - \sigma_2(\sigma_1(g)) = (\sigma_1(B)A - \sigma_2(A)B)g$$

and as the components of g form a basis we obtain

$$A(x^q)B(x) = B(x^p)A(x). \tag{3}$$

Our statement then follows from

Proposition 3. *Consider a system*

$$y(x^p) = A(x)y(x), \quad y(x^q) = B(x)y(x) \tag{4}$$

with multiplicatively independent positive integers p and q and $A(x), B(x) \in \mathrm{GL}_n(\mathbb{C}(x))$ satisfying the consistency condition (3). Suppose that $g(x) \in (\mathbb{C}[[x]][x^{-1}])^n$ is a formal vectorial solution. Then $g(x) \in \mathbb{C}(x)^n$.

Observe that we must actually have $n = 1$ in the proof of the Theorem because the components of $g(x)$ are $\mathbb{C}(x)$-linearly independent. ∎

The proof of Proposition 3 proceeds in three steps. We first prove that $g(x)$ converges in a neighborhood of 0. In the second step (the heart of the proof) we show that $g(x)$ can be extended analytically to a meromorphic function on $\mathbb{C}$ with only finitely many poles. Finally we prove that $g(x)$ has polynomial growth as $|x| \to \infty$ and therefore must be in $\mathbb{C}(x)^n$. We begin with the first step.

Lemma 4. *The series $g(x)$ is convergent in a neighborhood of* 0*.*

Proof. This is a special case of [5], Theorem 1-2, and could also be deduced from [8], section 4. For the convenience of the reader, we provide a short proof. To do that, we truncate $g(x)$ at a sufficiently high power of x to obtain $h(x) \in (\mathbb{C}[x][x^{-1}])^n$ and introduce $r(x) = h(x) - A(x)^{-1}h(x^p)$ and $\tilde{g}(x) = g(x) - h(x)$. Then we have

$$\tilde{g}(x) = A(x)^{-1}\tilde{g}(x^p) - r(x). \tag{5}$$

We denote the valuation of $A(x)^{-1}$ at the origin by $s \in \mathbb{Z}$ and introduce $\tilde{A}(x) = x^{-s}A(x)^{-1}$ which is holomorphic at the origin.

First choose $M \in \mathbb{N}$ such that $pM+s > M$ and $h(x)$ such that $g(x)-h(x)$ has at least valuation M. Then by (5), $r(x)$ also has at least valuation M. Now consider $R > 0$ such that $\tilde{A}(x)$ is holomorphic and bounded on $D(0,R)$. Then consider for positive $\rho < \min(R,1)$ the vector space E_ρ of all series $F(x) = \sum_{m=M}^{\infty} F_m x^m$ such that $\sum_{m=M}^{\infty} |F_m|\rho^m$ converges and define the norm $|F(x)|_\rho$ as this sum. Then E_ρ equipped with $|\ |_\rho$ is a Banach space and the existence of a unique solution of (5) in E_ρ for sufficiently small $\rho > 0$ follows from the Banach fixed-point theorem using that $|x^s F(x^p)|_\rho \leq \rho^{Mp+s-M}|F(x)|_\rho$ for $F(x) \in E_\rho$. Since any solution $y(x) \in x^M\mathbb{C}[[x]]$ of $y(x) = A(x)^{-1}y(x^p)$ must be zero, we have that $\tilde{g}(x)$ coincides with the solution in E_ρ. This proves the convergence of $\tilde{g}(x)$ and hence of $g(x)$. ∎

We now turn to the task of showing that $g(x)$ can be extended to a meromorphic function on $\mathbb{C}$. By (4), rewritten $g(x) = A(x)^{-1}g(x^p)$, the function g can only be extended analytically to a meromorphic function on the unit disk. As we want to extend it beyond the unit disk, we use the change of variables $x = e^t$, $u(t) = y(e^t)$ and obtain a system of q-difference equations

$$u(pt) = \bar{A}(t)u(t), \quad u(qt) = \bar{B}(t)u(t) \tag{6}$$

with $\bar{A}(t) = A(e^t)$, $\bar{B}(t) = B(e^t)$. It satisfies the consistency condition

$$\bar{A}(qt)\bar{B}(t) = \bar{B}(pt)\bar{A}(t). \tag{7}$$

Observe that $\bar{A}(t), \bar{B}(t)$ are not rational in t, but rational in e^t.

The heart of the proof of Proposition 3 lies in understanding the behavior of solutions of (6). We do this by first showing in Lemma 5 that there is a formal gauge transformation $u = G\,v$, $G \in \mathrm{GL}_n(\mathbb{C}\{t\}[t^{-1}])$, such that v satisfies a system with constant coefficients. We then show in Lemma 6 that the transformation matrix $G(t)$ and its inverse can be continued analytically to meromorphic functions on the t-plane. The "quotient" function $d(t) = G(t)^{-1}g(e^t)$ then satisfies a system with constant coefficients which can be solved explicitly. In this way, we show in Lemma 7 that $d(t)$ can be extended analytically to an entire function on the Riemann surface of $\log(t)$. Using these three lemmas, we show in Lemma 8 that $g(x)$ can be continued analytically to a meromorphic function on the x-plane.

Lemma 5. *There exists a convergent gauge transformation $u = G(t)\,v$, $G(t) \in \mathrm{GL}_n(\mathbb{C}\{t\}[t^{-1}])$, such that v satisfies*

$$v(pt) = A_1 v(t), \quad v(qt) = B_1 v(t) \tag{8}$$

where $A_1, B_1 \in \mathrm{GL}_n(\mathbb{C})$ commute.

Proof. Concerning the behavior at $t = 0$, it is known that there exists a formal gauge transformation $u = G\,z$, $G \in \mathrm{GL}_n(\mathbb{C}[[t^{1/s}]][t^{-1/s}])$, $s \in \mathbb{N}^*$, that reduces $u(pt) = \bar{A}(t)u(t)$ to a system $z(pt) = t^D A_1 z(t)$, where D is a diagonal matrix with entries in $\frac{1}{s}\mathbb{Z}$ and $A_1 \in \mathrm{GL}_n(\mathbb{C})$ such that any eigenvalue λ of A_1 satisfies $1 \le |\lambda| < |p|^{1/s}$, moreover D and A_1 commute. If we write $D = \mathrm{diag}(d_1 I_1, ..., d_r I_r)$ with distinct d_j and I_j identity matrices of an appropriate size, then $A_1 = \mathrm{diag}(A_1^1, ..., A_1^r)$ with diagonal blocks A_1^j of corresponding size. D and A_1 are essentially unique, *i.e.* except for a permutation of the diagonal blocks and passage from some A_1^j to a conjugate matrix. If D happens to be 0, then s can be chosen to be 1 and G is convergent (see [7], ch. 12, [2], [6]).

Now by the consistency condition (7), the gauge transformation $v = B(t)u$ transforms $u(pt) = \bar{A}(t)u(t)$ to $v(pt) = \bar{A}(qt)v(t)$. The gauge transformation $v = G(qt)w$ then transforms this system to $w(pt) = (qt)^D A_1 w(t)$. Now $(qt)^D A_1 = t^D\, q^D A_1$ and there is a diagonal matrix F with entries in $\frac{1}{s}\mathbb{Z}$ commuting with D and A_1 such that the gauge transformation $w = t^F \tilde{w}$ reduces the latter system to $\tilde{w}(pt) = t^D \tilde{A}_1 \tilde{w}(t)$, where $\tilde{A}_1 = p^{-F} q^D A_1$ has again eigenvalues with modulus in $[1, |p|^{1/s}[$. Now we write $\tilde{A}_1 = \mathrm{diag}(\tilde{A}_1^1, ..., \tilde{A}_1^r)$ and fix some $j \in \{1, ..., r\}$. If $a_1^j, ..., a_{r_j}^j$ are the eigenvalues of A_1^j then $p^{-f_j} q^{d_j} a_\ell^j$, $\ell = 1, ..., r_j$, are those of $\tilde{A}_1^j$. By the uniqueness of the reduced form, the mapping $t \mapsto p^{-f_j} q^{d_j} t$ induces a permutation of the eigenvalues of A_1^j. If we apply it several times, if necessary, we obtain the existence of some $\ell \in \{1, ..., r_j\}$ and of some positive integer k such that $p^{-kf_j} q^{kd_j} a_\ell^j = a_\ell^j$. Due to our condition on p and q this is only possible if $d_j = 0$. Thus we have proved that $D = 0$ and $t = 0$ is a so-called *regular singular point* of $u(pt) = \bar{A}(t)u(t)$.

We therefore obtain a matrix A_1 with eigenvalues λ in the annulus $1 \le |\lambda| < p$ and $G(t) \in \mathrm{GL}_n(\mathbb{C}\{t\}[t^{-1}])$ such that $u = G(t)v$ reduces the first equation of (6) to $v(pt) = A_1 v(t)$. This means

$$G(pt) = \bar{A}(t)G(t)A_1^{-1} \text{ for small } t. \tag{9}$$

Applying the same gauge transformation to the second equation of (6) yields an equation $v(qt) = \tilde{\bar{B}}(t)v(t)$ with some $\tilde{\bar{B}}(t) \in \mathrm{GL}_n(C\{t\}[t^{-1}])$. It satisfies the consistency condition $A_1\tilde{\bar{B}}(t) = \tilde{\bar{B}}(pt)A_1$. Now we expand $\tilde{\bar{B}}(t) = \sum_{m=m_0}^{\infty} C_m t^m$. The coefficients satisfy $A_1 C_m = C_m(p^m A_1)$, $m \ge m_0$. As A_1 and $p^m A_1$ have no common eigenvalue unless $m = 0$, we obtain that $\tilde{\bar{B}}(t) =: B_1$ is constant and commutes with A_1. We note the second equation satisfied by G

$$G(qt) = \bar{B}(t)G(t)B_1^{-1} \text{ for small } t. \tag{10}$$

∎

Lemma 6. *The functions $G(t)^{\pm 1}$ can be continued analytically to meromorphic functions on $\mathbb{C}$ and there exists $\delta > 0$ such that both can be continued analytically to the sectors $\{t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta\}$.*

Proof. Let $\mathcal{M}$ be the set of poles of $\bar{A}(t)^{\pm 1}$, *i.e.* the set of t such that e^t is a pole of $\bar{A}(x)$ or $\bar{A}(x)^{-1}$. Note that $\mathcal{M}$ is $2\pi i$-periodic, has no finite accumulation point and is contained in some vertical strip $\{t \in \mathbb{C} \mid -D < \operatorname{Re} t < D\}$. By (9), $G(t)^{\pm 1}$ can be continued analytically to $\mathbb{C}^* \setminus (\mathcal{M} \cdot p^{\mathbb{N}})$ and thus to meromorphic functions on $\mathbb{C}$ which we denote by the same name. By construction, $G(t)^{\pm 1}$ are also analytic in some punctured neighborhood of the origin. By the properties of $\mathcal{M}$, the infimum of the $|\operatorname{Re} t_1|$ on the set of all $t_1 \in \mathcal{M}$ having nonzero real part is a positive number. As $\mathcal{M}$ is contained in some vertical strip there exist sectors $\{t \in \mathbb{C}^* \mid \delta < \arg(\pm t) < 2\delta\}$ disjoint to $\mathcal{M}$ and hence to $\mathcal{M} \cdot p^{\mathbb{N}}$. Therefore $G(t)^{\pm 1}$ can be analytically continued to these sectors and the lemma is proved. ∎

Lemma 7. *The function $d(t) = G(t)^{-1} g(e^t)$ can be continued analytically to the Riemann surface of $\log(t)$.*

Proof. By Lemma 6 and because $g(x)$ is holomorphic in some punctured neighborhood of $x = 0$ by Lemma 4, $d(t)$ is defined and holomorphic for some sector $S = \{t \in \mathbb{C} \mid |t| > K,\ \pi + \delta < \arg t < \pi + 2\delta\}$. By (4), (9), and (10) it satisfies

$$d(pt) = A_1 d(t), \;\; d(qt) = B_1 d(t) \text{ for } t \in S. \tag{11}$$

To solve (11), consider a matrix L_1 commuting with B_1 such that $p^{L_1} = A_1$. Put $F(t) = t^{-L_1} d(t)$. Then

$$F(pt) = F(t), \;\; F(qt) = \tilde{B}_1 F(t) \text{ for } t \in S \tag{12}$$

where $\tilde{B}_1 = B_1 q^{-L_1}$. Thus $H(s) = F(e^s)$ is $\log(p)$-periodic on the half-strip $B = \{s \in \mathbb{C} \mid \operatorname{Re} s > \log(K),\ \pi + \delta < \operatorname{Im} s < \pi + 2\delta\}$ and can be expanded in a Fourier series. This implies that

$$F(t) = \sum_{\ell=-\infty}^{\infty} F_\ell \, t^{\frac{2\pi i}{\log(p)} \ell} \text{ for } t \in S. \tag{13}$$

The second equation of (12) yields conditions on the Fourier coefficients

$$F_\ell \exp\left(2\pi i \tfrac{\log(q)}{\log(p)} \ell\right) = \tilde{B}_1 F_\ell \text{ for } \ell \in \mathbb{Z}.$$

Therefore $F_\ell = 0$ unless $\exp\left(2\pi i \frac{\log(q)}{\log(p)} \ell\right)$ is an eigenvalue of $\tilde{B}_1$. Since p and q are multiplicatively independent, the quotient $\frac{\log(q)}{\log(p)}$ is irrational and hence $\exp\left(2\pi i \frac{\log(q)}{\log(p)}\right)$ is not a root of unity. Therefore all the numbers $\exp\left(2\pi i \frac{\log(q)}{\log(p)} \ell\right)$, $\ell \in \mathbb{Z}$ are different and only finitely many of them can be eigenvalues of $\tilde{B}_1$. This shows that the Fourier series (13) has finitely many terms and thus $F(t)$ can be analytically continued to the whole Riemann surface $\hat{\mathbb{C}}$ of $\log(t)$. The same holds for $d(t) = t^{L_1} F(t)$.
∎

Lemma 8. *The function $g(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ with finitely many poles.*

Remark: According to Theorem 4.2 of [8] (see also [4]), it is sufficient to show that $g(x)$ does not have the unit circle as a natural boundary and the rationality of $g(x)$ follows. We show how it follows naturally, in our context, that $g(x)$ can be continued analytically as a meromorphic function to all of $\mathbb{C}$ and, as well, that it has only finitely many poles. The rationality of $g(x)$ then follows as in [8] and [4] from a growth estimate (Lemma 9).

Proof. The function $h(t) = g(e^t)$ is holomorphic for t with large negative real part by Lemma 4 and $2\pi i$-periodic. Using Lemma 6 we conclude that $h(t) = G(t)d(t)$ can be analytically continued to a meromorphic function on $\hat{\mathbb{C}}$, in particular the point $t = 2\pi i$ is at most a pole of h. By its periodicity, this implies that $t = 0$ also is at most a pole of h and that it can be continued analytically to a meromorphic function on $\mathbb{C}$ which we denote by the same name.

Since $h(t) = g(e^t)$ for t with large negative real part, $h(t)$ is $2\pi i$-periodic for those values of t, hence also its analytic continuation to a meromorphic function on all of $\mathbb{C}$. This periodicity allows one to define a meromorphic function $\tilde{g}(x)$ on $\mathbb{C}\backslash\{0\}$ by $\tilde{g}(e^t) = h(t)$. As $\tilde{g}(x) = g(x)$ for small $|x| \neq 0$ by the construction of h, we have shown that $g(x)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ which will again be denoted by the same name.

The formula $h(t) = G(t)d(t)$ and Lemma 6 also imply that h is analytic in the sector $\tilde{S} = \{t \in \mathbb{C}^* \mid \delta < \arg t < 2\delta\}$. As this sector contains some half strip $\{t \in \mathbb{C} \mid \mathrm{Re}\, t > L, \mu \,\mathrm{Re}\, t < \mathrm{Im}\, t < \mu \,\mathrm{Re}\, t + 3\pi\}$ for some positive L, μ which has vertical width larger than 2π and h is $2\pi i$-periodic, its poles are contained in some vertical strip $\{t \in \mathbb{C} \mid -L < \mathrm{Re}\, t < L\}$. This implies that $g(x)$ has only a finite number of poles. ∎

The proof of Proposition 3 is completed once we have shown

Lemma 9. *The function $g(x)$ has polynomial growth as $|x| \to \infty$.*

Proof. This is shown in the proof of Theorem 4.2 in [8] (see also [4]). For the convenience of the reader, we reproduce it below.

Consider $r_0 > 1$ such that $g(x)$ and $A(x)$ are holomorphic on the annulus $|x| > r_0/2$. There are positive numbers K, M such that $|A(x)| \leq K|x|^M$ for $|x| \geq r_0$. Consider now the annuli

$$\mathcal{A}_j = \{x \in \mathbb{C} \mid r_0^{p^j} \leq |x| < r_0^{p^{j+1}}\},\ j = 0, 1, ...$$

covering the annulus $|x| \geq r_0$. Any $x \in \mathcal{A}_j$ can be written $x = \xi^{p^j}$ with some $\xi \in \mathcal{A}_0$. Then we estimate using (4) and the inequality for $|A(x)|$

$$|g(x)| = |g(\xi^{p^j})| \leq K^j \left(|\xi|^{p^{j-1}} \cdots |\xi|^p |\xi|\right)^M \max_{r_0 \leq |\xi| \leq r_0^p} |g(\xi)|.$$

Hence there is a positive constant L such that $|g(x)| \leq L\, K^j\, |x|^{\frac{M}{p-1}}$ for $x \in \mathcal{A}_j$. Assuming $\log(r_0) \geq 1$ without loss in generality, we find that $j \leq \log(\log(|x|))/\log(p)$ for $x \in \mathcal{A}_j$. Hence there exists $d > 0$ such that

$$|g(x)| \leq L\,(\log(|x|))^d\, |x|^{\frac{M}{p-1}} \text{ for } |x| > r_0.$$

∎

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