SOME APPLICATIONS OF LINEAR GROUPS TO DIFFERENTIAL EQUATIONS

By MICHAEL F. SINGER and MARVIN D. TRETKOFF*

Poincaré ([7], [8]) forged the link between the theory of n^{th} order homogeneous linear differential equations of Fuchsian class with $n > 2$ and the theory of linear groups, that is, subgroups of the general linear group, GL(n, **C**). The present paper and a complementary one by the first author ([11]) are an outgrowth of our attempt ([12]) to understand some of the implications of modern results about linear groups for equations of Fuchsian class. Namely, in ([12]) it is shown that a celebrated theorem of Tits ([13]) leads to a new classification of Fuchsian equations. In turn, that result leads to the following question which is addressed herein and in [11]:

When can the solutions to an n^{th} order homogeneous linear differential equation of Fuchsian class be expressed as combinations of solutions to lower order Fuchsian equations?

For example, suppose that we are given an n^{th} order Fuchsian equation with three singular points, $L(w) = 0$, $n > 2$, whose monodromy group is free of rank two. Then, in view of the ubiquity of the hypergeometric equation,

$$z(1 - z)w'' + [c - (a + b + 1)z]w' - abw = 0$$

it is natural to ask whether the solutions to $L(w) = 0$ can be expressed as combinations of hypergeometric functions. For example, if w_1 and w_2 form a basis for the solutions to the hypergeometric equation, then the complex vector space spanned by ${w_1}^2$, w_1w_2 and ${w_2}^2$ is the solution space of a third order equation of Fuchsian class. More complicated examples might involve combinations of hypergeometric functions defined by several choices of the parameters a, b and c.

In this connection, we prove (Theorem 2) that for every integer $n > 2$, there exists an n^{th} order equation of Fuchsian class which has three singularities and a free group of rank two as its monodromy group, but this

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equation cannot be solved by combinations of hypergeometric functions. Of course, it is necessary to state precisely what we mean when we say that an equation is "solvable by combinations of solutions of hypergeometric equations." We do this in the context of the differential Galois theory invented by Picard and Vessiot. Roughly speaking, we say that $L(w) = 0$ is *directly solvable by second order equations* if all of its solutions belong to a chain of differential fields which are obtained from one another successively by the adjunction of a basis for the solutions to a second order equation with coefficients in the preceding field. This definition allows us to adjoin integrals and exponentials of integrals, but not all algebraic functions. One compensation for this apparent limitation is the fact that we also establish (Theorem 3) the existence, for any $n \geq 3$, of an n^{th} order equation of Fuchsian class with three singularities, all of whose solutions are algebraic functions, but which cannot be solved directly by second order equations. Thus, *the algebraic functions arising as solutions to the hypergeometric equation do not generate all algebraic functions with three singularities.*

The proof of these facts is accomplished in two steps. First, we use the differential Galois theory to establish a necessary condition for equations which are directly solvable by second order equations. Next, we use the affirmative solution of the inverse problem for differential Galois theory over $\mathbf{C}(z)$, [14], to establish the existence of equations which do not satisfy this condition. Group-theoretic information, especially theorems of Platonov and Tits, plays an important role in both steps.

Finally, we note that, for the sake of simplicity, we have followed the custom (see, Poole, [9]) of discussing equations of Fuchsian class defined on the Riemann sphere. However, it is in fact the case that *our results are valid for equations of Fuchsian class which are defined on any compact Riemann surface.*

We begin by recalling a few of the concepts introduced by Poincaré ([7], [8]). Let

$$L(w) = a_0(z)w^{(n)} + \cdots + a_n(z)w = 0$$

be a differential equation of Fuchsian type on the Riemann sphere $\hat{\mathbf{C}}$. Thus, the coefficients $a_j(z)$ are rational functions and the singularities $z = b_1, \ldots, b_{t+1}$ of the equation are all regular singular points. Setting $Z = \hat{\mathbf{C}} - \{b_1, \ldots, b_{t+1}\}$ and selecting a base point $z = a$ in Z, we obtain a fundamental group $\pi_1(Z, a)$ which is free of rank t. We shall suppose that

$t \geq 2$ and that $b_{t+1} = \infty$. Thus, we may apply the uniformization theorem [10] to identify the universal covering of Z with the upper half, $\mathfrak{H}$, of the ζ-plane. Denoting the covering projection by $z = z(\zeta)$ and selecting a base point $\zeta = \alpha$ such that $a = \zeta(\alpha)$, we obtain a faithful representation of $\pi_1(Z, a)$ as a Fuchsian group of the first kind, Γ, acting on $\mathfrak{H}$ so that $\mathfrak{H}/\Gamma$ is conformally equivalent to Z.

Now, let $\mathbf{w}(z - a)$ be a column vector whose entries, $w_j(z - a), j = 1, \ldots, n$, form a basis for the space of solutions of $L(w) = 0$ in a neighborhood of $z = a$. Analytic continuation of $\mathbf{w}(z - a)$ to the end point $z = b$ of a path, σ, in Z yields a vector, $\mathbf{w}(z - b; \sigma)$, whose components form a basis for the solutions to $L(w) = 0$ in a neighborhood of $z = b$. It follows from the monodromy theorem that $\mathbf{w}(z - b; \sigma)$ only depends on the homotopy class, $[\sigma]$, of σ. Thus, analytic continuation along a loop in Z representing an element, γ, of Γ yields a column vector $\mathbf{w}(z - a; \gamma)$ of the form $m(\gamma)\mathbf{w}(z - a)$, where $m(\gamma)$ belongs to $\mathrm{GL}(n, \mathbf{C})$, the group of invertible complex matrices of degree n. Of course, $m(1)$ is the identity, so we shall continue to write $\mathbf{w}(z - a)$ instead of $\mathbf{w}(z - a; 1)$. Now, each path in Z beginning at $z = a$ has a unique lift to a path in $\mathfrak{H}$ beginning at $\zeta = \alpha$. Thus, the holomorphic vector defined in a neighborhood of $\zeta = \alpha$ by $\mathbf{w}(\zeta - \alpha) = \mathbf{w}(z(\zeta) - a)$ may be continued analytically to yield a holomorphic vector, $\mathbf{w}(\zeta)$, on $\mathfrak{H}$ such that

$$(*) \qquad \mathbf{w}(\zeta \cdot z) = m(\gamma)\mathbf{w}(\zeta), \; \gamma \in \Gamma, \; \zeta \in \mathfrak{H}$$

Moreover, since Γ is of the first kind, the real axis is the natural boundary of $\mathbf{w}(\zeta)$.

The correspondence $\gamma \rightarrow m(\gamma)$ affords a representation of Γ in $\mathrm{GL}(n, \mathbf{C})$ whose image and kernel will be denoted by M and K respectively. We call M the *monodromy group* of $L(w) = 0$, although it is only determined up to conjugacy in $\mathrm{GL}(n, \mathbf{C})$ because choices were made in the selection of the base points $z = a$, $\zeta = \alpha$ and the initial basis of the solution space at $z = a$. Poincaré, [8], referred to vectors satisfying (*) as *Zetafuchsian systems* of functions with respect to the pair of groups (Γ, M).

We now turn to the question of whether the solutions to an n^{th} order homogeneous linear differential equation, $L(w) = 0$, can be expressed in terms of the solutions to lower order equations. We shall suppose that the coefficients of $L(w)$ lie in a differential field, F, of characteristic zero with an algebraically closed field of constants, K. Our investigation is an application of group theoretic information by means of differential Galois theory. All of the information we shall require from the latter subject can be found in Kaplansky's admirable introduction to it, [6]. Here, we merely recall that if $w_1, \ldots, w_n$ are solutions to $L(w) = 0$ which are linearly independent over K, then we may form a *Picard-Vessiot extension*, $E = F\langle w_1, \ldots, w_n \rangle$, of F. This field plays a role which is analogous to that of a splitting field in ordinary Galois theory. Its elements are quotients of polynomials in the w_j and their derivatives. The automorphisms of E which commute with differentiation and which leave each element of F fixed form a group, $G(E/F)$, called the *differential Galois group* of E/F. Clearly, this group is a subgroup of $\mathrm{GL}(n, K)$; in fact, it is an algebraic matrix group (see, [6], Theorem 5.5), that is, it is a Zariski closed subgroup of $\mathrm{GL}(n, K)$.

The following definitions are basic to our investigation.

Definition 1. Suppose that $N \geq 2$ is an integer. We say that a homogeneous linear differential equation, $L(w) = 0$, with coefficients in F can be *solved directly by N^{th} order equations* provided there is a chain

$$F = F_0 \subset F_1 \subset \cdots \subset F_t$$

of differential fields such that:

(i) $E \subset F_t$, where E/F is a Picard-Vessiot extension associated with $L(w) = 0$

and

(ii) Each F_j/F_{j-1} is a Picard-Vessiot extension associated with a homogeneous linear differential equation, $L_j(w) = 0$, of degree at most N.

Remark. Our definition includes the situation where F_j is obtained from F_{j-1} by adjunction of either an integral of an element of F_{j-1} or the exponential of an integral of an element of F_{j-1}. Namely, adjunctions of this type yield, respectively, Picard-Vessiot extensions associated with the differential equations $w' - aw = 0$ or $aw'' - a'w = 0$, a in F_{j-1}. For more details, see [6], Chapter III, section 12.

The following group-theoretic notion is the key to our application of the differential Galois theory.

Definition 2. We say that a group G belongs to the class $\mathcal{G}_r$ provided that the orders of the non-abelian composition factors of its finite subgroups are at most r.

Thus, if S is a finite subgroup of a group, G, in $\mathcal{G}_r$ and if $1 = S_0 \lhd \cdots \lhd S_k = S$ is a composition series, then the order of S_i/S_{i-1} is at most r whenever this quotient is non-abelian. In particular, we see that abelian groups and solvable groups belong to $\mathcal{G}_r$ for all $r \geq 0$. On the other hand, if a group G contains a non-abelian simple group of order n as a subgroup, then it cannot belong to $\mathcal{G}_r$ if $r < n$.

The next Proposition summarizes certain properties of the groups we shall require in our application. As usual, the alternating and symmetric groups on n symbols will be denoted by A_n and S_n respectively. Moreover, SL(n, **C**) denotes the group of complex matrices of degree n with determinant one, and PSL(2, 7) denotes the group of fractional linear transformations of determinant 1 with coefficients in the field of seven elements.

PROPOSITION 1. (a) GL(1, **C**) $\in \mathcal{G}_0$, (b) GL(2, **C**) $\in \mathcal{G}_{60}$, (c) PSL(2, 7) $\notin \mathcal{G}_{60}$, (d) $A_n \notin \mathcal{G}_{60}$, $n > 5$, (e) SL(n, **C**) $\notin \mathcal{G}_{60}$, $n \geq 3$, (f) *the groups* PSL(2, 7) *and* A_n, $n > 5$, *can each be generated by two elements*, (g) PSL(2, 7) *has a faithful irreducible representation in* SL(3, **C**) (h) A_n *has a faithful irreducible representation in* SL($n - 1$, **C**), $n \geq 5$.

Proof. Since GL(1, **C**) is abelian, part (a) is immediate. Part (b) follows from the well-known fact (see, for example, Dornhoff, [3], page 144) that if S is a finite subgroup of GL(2, **C**), then either: (i) S has a normal abelian subgroup of index two, or, (ii) the quotient of S by its center is isomorphic to one of the three groups: A_4, A_5, S_4. Thus, S is either solvable or has a single non-abelian composition factor, A_5, of order sixty. Consequently, SL(2, **C**) belongs to $\mathcal{G}_{60}$.

Parts (c) and (d) are valid because the groups in question are finite simple groups whose orders exceed sixty.

Part (e) is a consequence of the fact that the groups occurring in parts (c) and (d) have faithful irreducible representations in SL(n, **C**), $n \geq 3$. In particular, PSL(2, 7) has such a representation, in SL(3, **C**). For example, see Burnside, [1], pages 309–311. The fact that A_n has a faithful irreducible representation in SL($n - 1$, **C**), $n \geq 5$ is proved in James and Kerber, [4], Theorem 2.5.15. These remarks also establish parts (g) and (h). Finally, we cite [2] as a convenient reference for part (f). This completes the proof of Proposition 1.

We shall now establish the properties of the class $\mathcal{G}_r$ which are important for our applications.

PROPOSITION 2. a) *If* G *belongs to* $\mathcal{G}_r$, *then any subgroup*, H, *of* G *also belongs to* $\mathcal{G}_r$.

b) *Suppose that N is a normal subgroup of G and suppose that both N and G/N belong to $\mathcal{G}_r$. Then, G belongs to $\mathcal{G}_r$.*

Proof. Part (a) is immediate from the definition of $\mathcal{G}_r$.

In order to prove (b), suppose that S is a finite subgroup of G and let $S^* \cong SN/N$ denote its image in G/N. According to our hypothesis, the orders of the non-abelian composition factors of S^* and $S \cap N$ are at most r. Now, it follows from the isomorphism theorems that the collection of composition factors of S is the union of the composition factors of $S \cap N$ and $S^* \cong SN/N \cong S/S \cap N$. Thus, our proposition is proved.

PROPOSITION 3. *Suppose that $\phi : G \to H$ is a K-morphism of Zariski closed subgroups of* GL(n, K), *K an algebraically closed field of characteristic zero. Let G^* denote the image of ϕ. If G belongs to $\mathcal{G}_r$, then G^* also belongs to $\mathcal{G}_r$.*

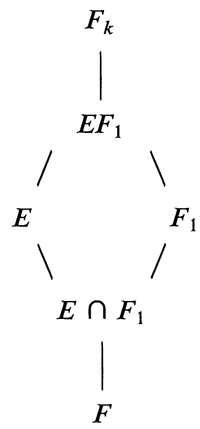
Proof. Suppose that S^* is a finite subgroup of G^* and set $S = \phi^{-1}(S^*)$. Since S^* is closed, so is S. Thus, we may apply a theorem of Platonov (see, for example, Wehrfritz, [15], Lemma 10.10) to conclude that $S = S^0 T$, where S^0 is the connected component of the identity in S and T is a finite subgroup of S. Since S^* is finite, S^0 belongs to the kernel of ϕ, and the restriction of ϕ to T is a surjection onto S^*. Now, the orders of the non-abelian composition factors of T do not exceed r because G belongs to $\mathcal{G}_r$; so it follows from the isomorphism theorem that the quotient group S^* has the same property. This completes the proof of Proposition 3.

The following theorem provides a necessary condition for equations which are directly solvable by second order equations.

THEOREM 1. *Suppose that $L(w) = 0$ is a homogeneous linear differential equation with coefficients in F, a differential field of characteristic zero with an algebraically closed field of constants. Moreover, suppose that $L(w) = 0$ is directly solvable by second order equations and let E/F denote the associated Picard-Vessiot extension. Then, the differential Galois group $G(E/F)$ belongs to $\mathcal{G}_{60}$.*

Proof. In accordance with Definition 1, E can be embedded in a chain of differential fields, $F = F_0 \subset \cdots \subset F_k$, such that each step F_i/F_{i-1} is a Picard-Vessiot extension associated with a first or second order equation. The proof proceeds by induction on the number of steps, k, in the chain. If $k = 0$, then $E = F$ and the Galois group $G(E/F) = 1$, so it belongs to $\mathcal{G}_{60}$. Next suppose that k is a positive integer and suppose that we have established Theorem 1 for Picard-Vessiot extensions which can be embedded in chains with $k - 1$ steps, each satisfying the condition specified by Definition 1.

Now, consider the diagram:



Here, as usual, EF_1 denotes the intersection of all the subfields of F_k containing E and F_1. Since E/F and F_1/F are Picard-Vessiot extensions, they are normal extensions ([6], Theorem 5.7). Therefore, according to the fundamental theorem of differential Galois theory ([6], Theorem 5.9), $G(F_k/E)$ and $G(F_k/F_1)$ are closed normal subgroups of $G(F_k/F)$. Consequently, $G(F_k/E) \cdot G(F_k/F_1)$ is also a closed normal subgroup of $G(F_k/F)$. Moreover, it is immediate from the definitions that the fixed field of this subgroup is $E \cap F_1$. Applying the fundamental theorem of differential Galois theory once again, we see that $E \cap F_1/F$ is a normal extension and that $G(E \cap F_1/F)$ is isomorphic to the quotient of $G(E/F)$ by $G(E/E \cap F_1)$. Consequently, it follows from Proposition 2 that we may prove that $G(E/F)$ belongs to $\mathcal{G}_{60}$ by proving that $G(E \cap F_1/F)$ and $G(E/E \cap F_1)$ both belong to $\mathcal{G}_{60}$.

In order to prove that $G(E/E \cap F_1)$ belongs to $\mathcal{G}_{60}$, we first observe that, according to Lemma 5.10 in [6], this group is isomorphic to $G(EF_1/F_1)$. Now, EF_1 is embedded in a chain, $F_1 \subset \cdots \subset F_k$, whose $k - 1$ steps each satisfy the requirements of Definition 1. Therefore, the inductive hypothesis assures us that $G(EF_1/F_1)$ belongs to $\mathcal{G}_{60}$, so $G(E/E \cap F_1)$ also belongs to $\mathcal{G}_{60}$.

Finally, we prove that $G(E \cap F_1/F)$ belongs to $\mathcal{G}_{60}$. According to the fundamental theorem of differential Galois theory, this group is isomorphic to the quotient of $G(F_1/F)$ by $G(F_1/E \cap F_1)$. Now, in view of parts (a) and (b) of Proposition 1 and part (a) of Proposition 2, $G(F_1/F)$ belongs to $\mathcal{G}_{60}$, so we may apply Proposition 3 to conclude that $G(E \cap F_1/F)$ also belongs to $\mathcal{G}_{60}$. This completes the proof of Theorem 1.

In order to establish the existence of differential equations with three singular points which are not directly solvable by second order equations, we reformulate the results of [14] in a form which is more convenient for our purpose.

THEOREM $(T - T)$. (a) *Suppose that* $z = b_1, \ldots, b_{t+1}, t \geq 2$, *are arbitrary points on* $\hat{\mathbf{C}}$ *and suppose that* $m_1, \ldots, m_{t+1}$ *are elements of* $\mathrm{GL}(n, \mathbf{C})$ *such that* $m_1 m_2 \cdots m_{t+1} = 1$. *Let* M *denote the group generated by the* m_j *and let* $\bar{M}$ *denote its Zariski closure. Finally, suppose that* $z_0 \neq b_j$, $1 \leq j \leq t + 1$. *Then there is an* n^{th} *order homogeneous linear differential equation of Fuchsian type*, $L(w) = 0$, *whose singularities are at* $z = b_1, \ldots, b_{t+1}$ *and which possesses a basis of solutions* $w_1, \ldots, w_n$ *at* z_0 *for which* M *is the monodromy group. Moreover,* $\bar{M}$ *is the differential Galois group of the Picard-Vessiot extension* $\mathbf{C}\langle z, w_1, \ldots, w_n\rangle$ *of* $\mathbf{C}(z)$.

(b) *Suppose that* G *is a Zariski closed subgroup of* $\mathrm{GL}(n, K)$, K *an algebraically closed field of characteristic zero. Then,* G *contains a finitely generated subgroup,* M, *whose closure* $\bar{M} = G$.

(c) *Suppose that* G *is a Zariski closed subgroup of* $\mathrm{GL}(n, \mathbf{C})$. *Then, there is a Picard-Vessiot extension,* E, *of* $\mathbf{C}(z)$ *whose differential Galois group is* G. *Moreover,* E *is defined by a differential equation of Fuchsian type on* $\hat{\mathbf{C}}$. [*Namely, according to part* (b), G *contains a Zariski dense subgroup,* M, *which is generated by finitely many elements* $m_1, \ldots, m_t$. *Setting* $m_{t+1} = (m_1 m_2 \cdots m_t)^{-1}$ *and applying part* (a), *we obtain part* (c)].

We may now prove the following result:

PROPOSITION 4. *There exists* n^{th} *order homogeneous linear differential equations of Fuchsian type on* $\hat{\mathbf{C}}$ *which are not directly solvable by second order equations.*

Proof. Let G be a Zariski closed subgroup of $\mathrm{GL}(n, \mathbf{C})$ which does not belong to $\mathcal{G}_{60}$. According to part (c) of Theorem $(T - T)$, G is the differential Galois group of an equation of Fuchsian type on $\hat{\mathbf{C}}$. This equation cannot be directly solvable by second order equations. Otherwise, we could apply Theorem 3 and assert that G belongs to $\mathcal{G}_{60}$. Thus, Proposition 4 is proved.

Although Proposition 4 asserts the existence of differential equations which are not directly solvable by second order equations, it does not provide any information about the number of singularities such equations possess. This is due to the fact that part (b) of Theorem $(T - T)$ makes no assertion about the number of elements required to generate a Zariski dense subgroup, M, of an arbitrary algebraic matrix group, G. However, if we suppose that G is semi-simple, then in contrast to the general situation, we have the following remarkable theorem of Tits, [13]:

THEOREM (Tits). *Suppose that G is a semi-simple algebraic matrix group defined over a field of characteristic zero. Then, G contains a Zariski dense subgroup, M, which, qua abstract group, is freely generated by two elements.*

We may now prove the following:

THEOREM 2. *Suppose $n \geq 3$ is an integer. Then, there is an n^{th} order homogeneous linear differential equation, $L(w) = 0$, of Fuchsian type on $\hat{\mathbf{C}}$ such that:*

(i) *$L(w) = 0$ has three singular points*
(ii) *The monodromy group of $L(w) = 0$ is free of rank two*
(iii) *$L(w) = 0$ cannot be solved directly by second order equations.*

Proof. Since $SL(n, \mathbf{C})$ is a simple algebraic matrix group, Tits' theorem assures us that it contains elements m_1 and m_2 which generate a free subgroup of rank two whose Zariski closure is $SL(n, \mathbf{C})$. Setting $m_3 = (m_1 m_2)^{-1}$ and selecting three arbitrary points, $z = b_1, b_2, b_3$, we may apply part (a) of Theorem $(T - T)$ to assert the existence of a differential equations with properties (i) and (ii). Since $SL(n, \mathbf{C})$ does not belong to $\mathcal{G}_{60}$, if $n \geq 3$ (Proposition 1, part (e)), this differential equation is not directly solvable by second order equations, so Theorem 2 is proved.

Now, finite subgroups of $GL(n, \mathbf{C})$ are obviously closed in the Zariski topology. Therefore, selecting three arbitrary points, $z = b_1, b_2, b_3$, and recalling (Proposition 1, part (h)) that $PSL(2, 7)$ and A_n, $n > 5$, can be generated by two elements and represented as subgroups of $SL(n, \mathbf{C})$, we can assert the existence of differential equations of Fuchsian type with these groups as both their monodromy groups and their differential Galois groups. These equations are not directly solvable by second order equations because their groups do not belong to $\mathcal{G}_{60}$ (Proposition 1, parts (d), (e)). Of course, all solutions of these equations are algebraic functions because their monodromy groups are finite and, consequently, a multi-valued function defined by a solution at $z_0 \neq b_1, b_3, b_3$ has but a finite number of branches at z_0. Finally, because the representations employed are irreducible, the differential equations in question are irreducible. That is, these equations are not composites of lower order homogeneous linear differential equations. The equivalence of the irreducibility of a homogeneous linear differential equation and the irreducibility of its monodromy group was apparently first proved by C. Jordan [5]. Corresponding statements can be made about the differential Galois group; see Kaplansky [6], page 40, for references.

We summarize our discussion in the following:

THEOREM 3. *Suppose $n \geq 3$ is an integer. Then there is an n^{th} order homogeneous linear differential equation of Fuchsian type, $L(w) = 0$, on $\hat{\mathbf{C}}$ such that:*

(i) *$L(w) = 0$ has three singularities*
(ii) *$L(w) = 0$ is not directly solvable by second order equations*
(iii) *All the solutions of $L(w) = 0$ are algebraic functions*
(iv) *$L(w) = 0$ is irreducible.*

Finally, we note that Theorems 1, 2 and 3 have analogues wherein the phrase "directly solvable by second order equations" is replaced by "directly solvable by N^{th} order equations" for any integer $N \geq 2$. To see this, we note the following corollary of a well known theorem of Jordan (see, [3], Theorem 30.4). There is an integer valued function, $k = k(N)$, such that the orders of the non-abelian composition factors of the finite subgroups of $\mathrm{GL}(N, \mathbf{C})$, $N > 1$, are at most $k(N)$. Now, the proof of Theorem 1 remains valid if we replace 60 with $k(N)$. Once again, for n sufficiently large, $\mathrm{SL}(n, \mathbf{C})$ does not belong to $\mathcal{G}_{k(N)}$. Thus, we may formulate the following:

THEOREM 4. *Suppose $N \geq 2$ is an integer. Then, there is an integer $I(N)$ and, for each integer $n \geq I(N)$, an n^{th} order homogeneous linear differential equation, $L(w) = 0$, of Fuchsian type on $\hat{\mathbf{C}}$ such that:*

(i) *$L(w) = 0$ has three singular points*
(ii) *The monodromy group of $L(w) = 0$ is free of rank two*
(iii) *$L(w) = 0$ cannot be solved directly by N^{th} order equations.*

Similarly, we have:

THEOREM 5. *Suppose that $N \geq 2$ is an integer. Then, there is an integer $I(N)$ and, for each integer $n \geq I(N)$, an n^{th} order homogeneous linear differential equation, $L(w) = 0$, of Fuchsian type on $\hat{\mathbf{C}}$ such that:*

(i) *$L(w) = 0$ has three singularities*
(ii) *$L(w) = 0$ is not directly solvable by N^{th} order equations*
(iii) *All the solutions of $L(w) = 0$ are algebraic functions*
(iv) *$L(w) = 0$ is irreducible.*

NORTH CAROLINA STATE UNIVERSITY
THE INSTITUTE FOR ADVANCED STUDY

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