

# MONODROMY GROUPS OF PARAMETERIZED LINEAR DIFFERENTIAL EQUATIONS WITH REGULAR SINGULARITIES

CLAUDE MITSCHI AND MICHAEL F. SINGER

ABSTRACT. We study the notion of regular singularities for parameterized complex ordinary linear differential systems, prove an analogue of the Schlesinger theorem for systems with regular singularities and solve both a parameterized version of the weak Riemann-Hilbert Problem and a special case of the inverse problem in parameterized Picard-Vessiot theory.

## 1. INTRODUCTION

Parameterized families of linear differential systems with regular singular points arise in the study of isomonodromic as well as monodromy evolving deformations and their relation to the equations of mathematical physics ([3], [9, 10], [17], [18], [28], [24, 25], [22]) . In this paper we address some fundamental questions concerning the monodromy groups of such systems. We begin, in Section 2, by studying equivalent definitions of regular singular parameterized linear differential systems and proving bounds on the growth of solutions of such systems. In Section 3, we show that the parameterized monodromy matrices of such a system belong to the associated parameterized Picard-Vessiot group. In Section 4, we prove an analogue of the Schlesinger theorem for systems with regular singularities. Our result states that for parameterized systems with only regular singularities, the parameterized monodromy matrices generate a Kolchin-dense subgroup of the parameterized Picard-Vessiot group. In Section 5, we solve a parameterized version of the weak Riemann-Hilbert Problem and a special case of the inverse problem in parameterized Picard-Vessiot theory.

## 2. PARAMETERIZED REGULAR SINGULARITIES

Let  $\mathcal{U}$  be an open connected subset of  $\mathbb{C}^r$  with  $0 \in \mathcal{U}$ , and let  $\mathcal{O}_{\mathcal{U}}$  be the ring of analytic functions on  $\mathcal{U}$  of a variable  $t$ . Let  $\alpha \in \mathcal{O}_{\mathcal{U}}$  and assume  $\alpha(0) = 0$ . We will denote

- (1) by  $\mathcal{O}_{\mathcal{U}}((x - \alpha(t)))$  the ring of formal Laurent series in powers of  $x - \alpha(t)$  with coefficients in  $\mathcal{O}_{\mathcal{U}}$ , that is, elements

$$f(x, t) = \sum_{i \geq m} a_i(t)(x - \alpha(t))^i$$

where  $m \in \mathbb{Z}$  is independent of  $t$ ,

- (2) by  $\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\})$  the ring of those  $f(x, t) \in \mathcal{O}_{\mathcal{U}}((x - \alpha(t)))$  that, for each fixed  $t \in \mathcal{U}$ , converge for  $0 < |x - \alpha(t)| < R_t$ , for some  $R_t > 0$ .

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**Lemma 2.1.** Let  $f(x, t) \in \mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\})$  and let  $\mathcal{N} \subset \mathcal{U}$  be a compact neighborhood of 0. Then there is  $R > 0$  such that the series  $f(x, t)$  converges for all  $t \in \mathcal{N}$  and  $0 < |x - \alpha(t)| < R$ .

*Proof.* For each  $t \in \mathcal{U}$  we may assume that  $R_t$  is maximal, possibly infinite. For finite  $R_t$ , let  $\Gamma(\alpha(t), R_t)$  and  $D(\alpha(t), R_t)$  denote the circle and open disk respectively, with center  $\alpha(t)$  and radius  $R_t$ . If  $R_{t_0} = \infty$  for some  $t_0 \in \mathcal{U}$  then clearly  $R_t = \infty$  for all  $t \in \mathcal{U}$ . Assuming this is not the case,  $R_t$  is a continuous function of  $t$ . To prove this, fix  $t_0 \in \mathcal{U}$ , and a neighborhood  $u(t_0)$  of  $t_0$  in  $\mathcal{U}$  such that  $\alpha(t) \in D(\alpha(t_0), R_{t_0})$  for all  $t \in u(t_0)$ . If  $t \in u(t_0)$ , the circles  $\Gamma(\alpha(t_0), R_{t_0})$  and  $\Gamma(\alpha(t), R_t)$  either are equal, or intersect in two points, or are inner tangent. A simple geometric argument shows that

$$|R(t) - R(t_0)| \leq |\alpha(t) - \alpha(t_0)|$$

for all  $t \in u(t_0)$ , and the continuity of  $R$  follows from the continuity of  $\alpha$ . Since  $\mathcal{N}$  is compact, and  $R_t > 0$  for all  $t \in \mathcal{N}$ , the function  $R_t$  (possibly infinite) has a lower bound  $R > 0$  on  $\mathcal{N}$ .  $\square$

Consider a parameterized linear differential equation

$$(2.1) \quad \frac{\partial Y}{\partial x} = AY$$

where  $A \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$ . Note that  $A$  may be written as

$$A(x, t) = \frac{A_{-m}(t)}{(x - \alpha(t))^m} + \frac{A_{-m+1}(t)}{(x - \alpha(t))^{m-1}} + \dots = \sum_{i \geq -m} (x - \alpha(t))^i A_i(t)$$

where  $A_i(t) \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$  for all  $i \geq -m$ , and  $m \in \mathbb{N}$  does not depend on  $t$ .

**Definition 2.2.** Two equations

$$\frac{\partial Y}{\partial x} = AY \quad \text{and} \quad \frac{\partial Y}{\partial x} = BY,$$

with  $A, B \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$ , are *equivalent* if there exists  $P \in \text{GL}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$  such that

$$B = \frac{\partial P}{\partial x} P^{-1} + PAP^{-1},$$

that is, if  $Y$  satisfies the first equation, then  $PY$  satisfies the second.

**Definition 2.3.** With notation as before,

- (1) Equation (2.1) has *simple singular points near 0* if  $m = 1$  and  $A_{-1} \neq 0$  as an element of  $\mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$ ,
- (2) Equation (2.1) has *parameterized regular singular points near 0* if it is equivalent to an equation with simple singular points near 0.

Let  $\delta = (x - \alpha(t)) \frac{\partial}{\partial x}$ . If Equation (2.1) has simple singular points near 0 we may also write it as

$$(2.2) \quad \delta Y = \left( \sum_{i \geq 0} (x - \alpha(t))^i A_i(t) \right) Y$$

with (renamed)  $A_i \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(\{x - \alpha(t)\}))$  and  $A_0 \neq 0$ . We will show that such an equation is equivalent to an equation of a simpler form. The proof is a slight modification of the similar one for non-parameterized equations in [27].

**Proposition 2.4.** Assume that in Equation (2.2) no eigenvalues of  $A_0(0)$  differ by positive integers. Then there is an open connected subset  $\mathcal{U}'$  of  $\mathcal{U}$  and matrices  $P_i \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$  such that

- 1) the substitution  $Z = PY$ , with

$$P(x, t) = I + \sum_{i \geq 1} (x - \alpha(t))^i P_i(t),$$

transforms Equation (2.2) into

$$\delta Z = A_0(t)Z,$$

- 2) the series  $P(x, t)$  converges for  $(x, t) \in D(\alpha(t), R) \times \mathcal{U}'$ , for some  $R > 0$  which does not depend on  $t$ .

*Proof.* We begin by noting that since no eigenvalues of  $A_0(0)$  differ by positive integers, there is a neighborhood  $\mathcal{U}' \subset \mathcal{U}$  of 0 such that for  $t \in \mathcal{U}'$ , no eigenvalues of  $A_0(t)$  differ by positive integers. We will now follow the proof of Proposition 3.12 of ([27], p. 64). We wish to construct a matrix  $P$ ,

$$P(x, t) = I + \sum_{i \geq 1} (x - \alpha(t))^i P_i(t)$$

such that

$$A_0(t)P(x, t) = P(x, t)A(x, t) + \delta P(x, t).$$

Comparing powers of  $x - \alpha(t)$  we see that

$$(2.3) \quad A_0 P_i - P_i (A_0 + iI) = A_i + A_{i-1} P_1 + \dots + A_1 P_{i-1}$$

for all  $i \geq 1$  (with  $P_0 = I$ ). As noted above, for fixed  $t \in \mathcal{U}'$  no eigenvalues of  $A_0(t)$  differ by positive integers. Therefore, for  $t \in \mathcal{U}'$ , the map

$$X \mapsto A_0(t)X - X(A_0(t) + iI)$$

is an isomorphism on  $\mathfrak{gl}_n(\mathbb{C})$  and the matrix  $M(t)$  representing this  $\mathbb{C}$ -linear map has a nonzero determinant. This implies that the entries of  $M(t)^{-1}$  are analytic on  $\mathcal{U}'$  and that we can solve Equation (2.3) to find matrices  $P_i(t)$  whose entries are analytic on  $\mathcal{U}'$ .

We now turn to the statement concerning convergence. The formal power series  $P(x, t)$  (in powers of  $x - \alpha(t)$ ) satisfies the differential equation

$$\delta P(x, t) = A_0(t)P(x, t) - P(x, t)A(x, t).$$

For each fixed value of  $t$  this is a differential equation with a simple (Fuchsian) singularity. Lemma 3.9.2 of [33] or the proof of Lemma 3.42 of [27] implies that  $P(x, t)$  has a radius of convergence at least as large as that for  $A(x, t)$ . We may assume that  $\overline{\mathcal{U}'}$  is compact with  $\overline{\mathcal{U}'} \subset \mathcal{U}$  and hence, by Lemma 2.1 above, that there is an  $R > 0$  such that  $A(x, t)$  converges for all  $t \in \mathcal{U}'$  and  $0 < |x - \alpha(t)| < R$ , which ends the proof of 2).  $\square$

Let us deduce from Proposition 2.4 a slightly weaker result without the hypothesis on  $A_0(0)$ .

**Corollary 2.5.** Consider the general equation (2.2). Then there exists

- a constant matrix  $C \in \mathrm{GL}_n(\mathbb{C})$ ,

- an  $n \times n$  matrix

$$S = \begin{pmatrix} (x - \alpha(t))^{r_1} I_1 & 0 & \dots & 0 \\ 0 & (x - \alpha(t))^{r_2} I_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & (x - \alpha(t))^{r_s} I_s \end{pmatrix}$$

where for some  $s$  the  $r_i$ ,  $i = 1, \dots, s$ , are nonnegative integers and the  $I_i$  are identity matrices of various sizes,

- an open connected subset  $\mathcal{U}'$  of  $\mathcal{U}$  and matrices  $P_i \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$ ,

such that

- 1) the substitution  $Z = PSCY$ , with

$$P(x, t) = I + \sum_{i \geq 1} (x - \alpha(t))^i P_i(t),$$

transforms (2.2) into

$$(2.4) \quad \delta Z = \tilde{A}(t)Z$$

where  $\tilde{A} \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$ ,

- 2) the series  $P(x, t)$  converges for  $(x, t) \in D(\alpha(t), R) \times \mathcal{U}'$  for some  $R > 0$  which does not depend on  $t$ .

*Proof.* We will perform a *shearing transformation* to replace Equation (2.2) by an equation satisfying the condition of Proposition 2.4 on  $A_0(0)$ , as follows. Let  $c_1, \dots, c_s$  be the distinct eigenvalues of  $A_0(0)$  and assume that  $c_2 - c_1 = m$ , a positive integer. Replacing  $Y$  by  $CY$  for some  $C \in \mathrm{GL}_n(\mathbb{C})$ , if needed, we can assume that

$$A_0(0) = \begin{pmatrix} A_{0,1} & 0 & \dots & 0 \\ 0 & A_{0,2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_{0,s} \end{pmatrix}$$

where  $A_{0,i}$ , for  $i = 1, \dots, s$ , is a matrix in Jordan normal form with eigenvalue  $c_i$ . Let

$$T = \begin{pmatrix} (x - \alpha(t))^m I_1 & 0 & \dots & 0 \\ 0 & I_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I_s \end{pmatrix}$$

where each  $I_i$  is an identity matrix of the same size as  $A_{0,i}$ . The substitution  $Z = TY$  transforms Equation (2.2) into an equation

$$\delta Z = \left( \sum_{i \geq 0} (x - \alpha(t))^i \tilde{A}_i \right) Z$$

where the eigenvalues of  $\tilde{A}_0(0)$  are  $c_1 + m, c_2, \dots, c_s$ , that is,  $\tilde{A}_0(0)$  has fewer eigenvalues differing by positive integers. By induction, one constructs a matrix  $S$  such that the transform of Equation (2.2) via  $Z = SCY$  satisfies the condition of Proposition 2.4, whose conclusion ends the proof.  $\square$

The following corollary shows that our definition of a parameterized regular singularity yields, as in the non-parameterized case, solutions that have moderate growth in the neighborhood of this moving singularity.

**Corollary 2.6.** Assume that Equation (2.1) has regular singular points near 0. Then there is an open connected subset  $\mathcal{U}'$  of  $\mathcal{U}$  such that

- 1) Equation (2.1) has a solution  $Y$  of the form

$$(2.5) \quad Y(x, t) = \left( \sum_{i \geq i_0} (x - \alpha(t))^i Q_i(t) \right) (x - \alpha(t))^{\tilde{A}(t)}$$

with  $\tilde{A} \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$  and  $Q_i \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}'})$  for all  $i \geq i_0$ ,

- 2) for any  $r$ -tuple  $(m_1, \dots, m_r)$  of nonnegative integers there is an integer  $N$  such that for any fixed  $t \in \mathcal{U}'$  and any sector  $\mathcal{S}_t$  from  $\alpha(t)$  in the complex plane, of opening less than  $2\pi$ ,

$$\lim_{\substack{x \rightarrow \alpha(t) \\ x \in \mathcal{S}_t}} (x - \alpha(t))^N \frac{\partial^{m_1 + \dots + m_r} Y(x, t)}{\partial^{m_1} t_1 \dots \partial^{m_r} t_r} = 0.$$

*Proof.* Under our assumptions, Equation (2.1) is equivalent to an equation with simple singularities near 0. We shall prove the conclusion for this new equation and one easily sees that it holds for the original equation as well.

Assume Equation (2.1) has simple singular points near 0. Corollary 2.5 implies that there are  $C, S$  and  $P$  as described such that  $Y$  is a solution of Equation (2.2) whenever  $Z = PSCY$  is a solution of Equation (2.4). This shows that

$$Y = (PSC)^{-1} (x - \alpha(t))^{\tilde{A}(t)}$$

is a solution of the desired form (2.5). Differentiating (2.5) yields a form that satisfies the conclusion of 2).  $\square$

Solutions of parameterized differential equations with irregular singularities have been studied in [1] and [29]. Assuming 0 is a (non-moving) irregular singularity, these authors gave a “well-behaviour” condition on the exponential part of a formal solution in the usual form

$$\hat{Y}(z) = \hat{H}(z) z^J e^Q$$

to ensure that the coefficients of the formal series  $\hat{H}(z)$  depend analytically on the multiparameter.

### 3. PARAMETERIZED MONODROMY

**3.1. Classical Picard-Vessiot theory and monodromy.** Consider a differential equation

$$\frac{dY}{dx} = A(x)Y$$

where  $A \in \mathfrak{gl}_n(\mathbb{Q}(x))$ . To apply differential Galois theory to this equation we need to work over an algebraically closed field containing  $\mathbb{Q}$ , for example  $\bar{\mathbb{Q}}$ , the algebraic closure of  $\mathbb{Q}$ . But when we talk about monodromy matrices and want to say that the monodromy matrices lie in the differential Galois group, or Picard-Vessiot group, we need to account for the possibility that these matrices have transcendental entries. To deal with this, we usually consider the Picard-Vessiot theory over  $\mathbb{C}(x)$ , but we might use any algebraically closed field  $k_0$  containing the entries of the monodromy

matrices.

This raises the question: does the group change when we go to a bigger field of constants? The answer is given by the following proposition. We note that Propositions 3.1 and 3.3 below can be stated in far greater generality than stated here (see [13], Theorem 9.10 for an approach via Tannakian categories and also [11], pp. 80-81 for similar results concerning difference equations) yet we present a proof here using simple tools that allows us to prove this result in our restricted setting.

If not otherwise specified, the derivation will be denoted by  $(\ )'$ . We will write PV-group and PV-extension, for short, for the Picard-Vessiot group and Picard-Vessiot extension respectively.

**Proposition 3.1.** Let  $C_0 \subset C_1$  be algebraically closed fields and  $k_0 = C_0(x), k_1 = C_1(x)$  be differential fields where  $c' = 0$  for all  $c \in C_1$  and  $x' = 1$ . Let

$$(3.1) \quad Y' = AY$$

be a differential equation with  $A \in \mathfrak{gl}_n(k_0)$ . If  $G(C_0) \subset \mathrm{GL}_n(C_0)$  is the PV-group over  $k_0$  of Equation (3.1) with respect to some fundamental solution, where  $G$  is a linear algebraic group defined over  $C_0$ , then  $G(C_1)$  is the PV-group of (3.1) over  $k_1$ , with respect to some fundamental solution.

*Proof.* Equation (3.1) has a regular point in  $C_0$  and we shall assume this is  $x = 0$ . We can think of  $C_i(x)$  as a subfield of the field (of formal Laurent series)  $C_i((x))$  for  $i = 0, 1$ . Since 0 is a regular point, Equation (3.1) has a solution  $Z_0 \in \mathrm{GL}_n(C_0[[x]]) \subset \mathrm{GL}_n(C_0((x)))$  (note that this solution is found by substituting  $Z_0 = Z_{0,0} + Z_{0,1}x + Z_{0,2}x^2 + \dots$  into (3.1) and equating powers of  $x$ . Assuming  $Z_{0,0} = I_n$  ensures that  $Z_0 \in \mathrm{GL}_n(C_0[[x]])$ ). Since the differential field  $K_0 = k_0(Z_0) \subset k_0((x))$  has no new constants, it is a PV-extension of  $k_0$  for Equation (3.1). Similarly,  $K_1 = k_1(Z_0)$  is a PV-extension of  $k_1$  for Equation (3.1).

Let  $R_0 = k_0[Z_0]$  and  $R_1 = k_1[Z_0]$ . We may write  $R_0 = k_0[Y]/\mathcal{I}_0$  and  $R_1 = k_1[Y]/\mathcal{I}_1$  where  $Y$  is a set of  $n^2$  indeterminates and  $\mathcal{I}_0$  and  $\mathcal{I}_1$  are ideals in their respective rings. Let us prove that  $\mathcal{I}_1 = k_1 \cdot \mathcal{I}_0$ . Clearly,  $k_1 \cdot \mathcal{I}_0 \subset \mathcal{I}_1$ . To show the other inclusion, let  $P \in \mathcal{I}_1$ . By clearing denominators, we may assume that  $P \in C_1[x][Y]$ . Let us write  $P = \sum d_i P_i$  where  $\{d_i\}$  is a  $C_0$ -basis of  $C_1$  and  $P_i \in C_0[x][Y]$ . Note that the  $d_i$  are linearly independent over  $C_0[[x]]$  as well. Substituting  $Z_0$  for  $Y$  in  $P$ , we have that  $P(Z_0) = \sum d_i P_i(Z_0) = 0$ . Therefore  $P_i \in \mathcal{I}_0$  for each  $i$  and so  $P \in k_1 \cdot \mathcal{I}_0$ .

For  $i = 0, 1$ , the PV-group of  $K_i$  over  $k_i$  consists of the matrices  $B$  in  $\mathrm{GL}_n(C_i)$  such that  $Z_1 B$  is again a zero of  $\mathcal{I}_i$ . Let  $\mathcal{I}_0 = (f_1, \dots, f_m)$  be generated in  $k_0[Y]$  by polynomials  $f_i \in k_0[Y]$ . The PV-group of  $K_0$  over  $k_0$  then consists of all  $B \in \mathrm{GL}_n(C_0)$  such that  $f_j(Z_0 B) = 0$  for  $j = 1, \dots, m$ . We may consider  $f_j(Z_0 B)$  as an element of  $C_0((x))$  and write  $f_j(Z_0 B) = \sum_{\ell \geq \ell_0} f_{\ell,j}(B) x^\ell$  where  $f_{\ell,j}(B)$  is a polynomial in the entries of  $B$  with coefficients in  $C_0$ . The PV-group of  $K_0$  over  $k_0$  is therefore defined as

$$G_0 = \{B \in \mathrm{GL}_n(C_0) \mid f_{\ell,j}(B) = 0, \text{ for } j = 1, \dots, m \text{ and } \ell \geq \ell_0, \}.$$

Since  $\mathcal{I}_1 = k_1 \cdot \mathcal{I}_0$ , we have  $\mathcal{I}_1 = (f_1, \dots, f_m)$  in  $k_1[Y]$ , and the PV-group of  $K_1$  over  $k_1$  is also defined as  $G_1 = \{B \in \mathrm{GL}_n(C_1) \mid f_j(Z_0 B) = 0 \text{ for } j = 1, \dots, m\}$  and therefore as

$$G_1 = \{B \in \mathrm{GL}_n(C_1) \mid f_{\ell,j}(B) = 0, \text{ for } j = 1, \dots, m \text{ and } \ell \geq \ell_0, \}.$$

This means that if  $G$  is a linear algebraic group defined over  $C_0$  by  $\{f_{\ell,j}(B) = 0\}$  then the PV-group of  $K_i$  over  $k_i$  is  $G(C_i)$  for  $i = 0, 1$ . To end the proof, note that the linear algebraic group  $G$  defined over  $C_0$  is uniquely determined by the group  $G(C_0)$  of its  $C_0$ -points.  $\square$

**Corollary 3.2.** Assume in Equation (3.1) that  $A \in \mathfrak{gl}_n(C_0(x))$  where  $C_0$  is some algebraically closed subfield of  $\mathbb{C}$ . Assuming 0 is a regular point, let us fix it as the base-point of  $\pi_1(\mathbb{CP}^1 \setminus \mathcal{S})$ , where  $\mathcal{S}$  is the set of singular points of (3.1) on  $\mathbb{CP}^1$ . Let  $G(C_0)$  be the PV-group of (3.1) over  $C_0(x)$ , where  $G$  is a linear algebraic group defined over  $C_0$ . If  $C_1$  is any algebraically closed subfield of  $\mathbb{C}$  containing  $C_0$  and the entries of the monodromy matrices, then the monodromy matrices are elements of the PV-group  $G(C_1)$  of (3.1) over  $C_1(x)$ .

*Proof.* With notation from the proof of Proposition 3.1, note that the matrix  $Z_0$  has entries that are *convergent* in some neighborhood of 0 since at a regular point formal solutions are convergent. Let  $M$  be a monodromy matrix with respect to  $Z_0$  corresponding to an element  $[\gamma]$  of  $\pi_1(\mathbb{CP}^1 \setminus \mathcal{S})$  and let  $P \in \mathcal{I}_1$ , that is,  $P(Z_0) = 0$ . We again may assume that  $P \in C_1[x][Y]$ . To show that  $M \in G(C_1)$  it is enough to show that  $P(Z_0B) = 0$ . Note that we can analytically continue  $P(Z_0)$  around  $\gamma$  and the result will be  $P(Z_0B)$ . Since  $P(Z_0) = 0$ , we have  $P(Z_0B) = 0$  as well.  $\square$

**3.2. Parametrized Picard-Vessiot theory and monodromy.** In the parametrized Picard-Vessiot theory (PPV-theory for short) one has a similar issue: the equation may have coefficients that lie in one differentially closed field while the parameterized monodromy matrices have entries that lie in a larger differentially closed field. For general definitions and facts about PPV-theory we refer to [8] and [20].

We will first prove a result similar to Proposition 3.1 for parameterized Picard-Vessiot extensions (PPV-extensions for short). For simplicity of notation, we shall consider equations of the form

$$(3.2) \quad \frac{\partial Y}{\partial x} = A(x, t)Y$$

where  $A(x, t) \in \mathfrak{gl}_n(\mathcal{O}_{\mathcal{U}}(x))$ , for some domain  $\mathcal{U} \in \mathbb{C}^r$ . We shall denote differentiation with respect to  $x, t_1, \dots, t_r$  by  $\partial_x, \partial_{t_1}, \dots, \partial_{t_r}$  respectively, and let  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$  and  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$ . Let  $C$  be a differentially closed  $\Delta_t$ -extension of some field of functions that are analytic on some domain of  $\mathbb{C}^r$ , and let  $\partial_{t_i}$  denote for each  $i$  the derivation extending  $\partial_{t_i}$ . We define the differential  $\Delta_t$ -field  $k = C(x)$ , where  $x$  is an indeterminate over  $C$ , by letting  $\partial_x(x) = 1, \partial_{t_i}(x) = 0$  for each  $i$  and  $\partial_x(c) = 0$  for all  $c \in C$ . For the definition of a differentially closed field, see ([8], Definition 3.2).

PPV-extensions of  $k$  for Equation (3.2) are of the form  $K = k\langle Z \rangle$  where  $K$  has no new  $\partial_x$ -constants and  $Z$  is a fundamental solution for (3.2). The brackets  $\langle \dots \rangle$  denote the fact that  $K$  is generated, as a  $\Delta_t$ -field, by the entries of  $Z$ , that is,  $K = k(Z, \partial_{t_1}Z, \dots, \partial_{t_r}Z, \dots, \partial_{t_1}^{\alpha_1} \dots \partial_{t_r}^{\alpha_r}Z, \dots)$ . Note that this is automatically a  $\partial_x$ -field as well, with  $\partial_x Z = AZ, \partial_x(\partial_{t_i}Z) = (\partial_{t_i}A)Z + A\partial_{t_i}Z$  for each  $i$ , etc..

**Proposition 3.3.** Let  $C_0 \subset C_1$  be differentially closed  $\Delta_t$ -fields and let  $k_0 = C_0(x), k_1 = C_1(x)$  be  $\Delta$ -fields as above. Let

$$(3.3) \quad \partial_x Y = AY$$

be a differential equation with  $A \in \mathfrak{gl}_n(k_0)$ . If  $G(C_0) \subset \mathrm{GL}_n(C_0)$  is the PPV-group over  $k_0$  of Equation (3.3) with respect to some fundamental solution, where  $G$  is a linear differential algebraic group defined over the differential  $\Delta_t$ -field  $C_0$ , then  $G(C_1)$  is the PPV-group over  $k_1$  of (3.3) with respect to some fundamental solution.

*Proof.* The proof is basically the same as in the non-parameterized case. Equation (3.3) has a regular point in  $C_0$  and we shall assume this is  $x = 0$ . We can think of  $C_i(x)$  as a subfield of the field (of formal Laurent series)  $C_i((x))$  for  $i = 0, 1$ . Since 0 is a regular point, Equation (3.3) has a solution  $Z_0 \in \mathfrak{gl}_n(C_0((x)))$ . Assuming (with previous notation  $Z_0 = \sum Z_{0,i}x^i$ ) that  $Z_{0,0} = I_n$ , ensures that  $Z_0 \in \mathrm{GL}_n(C_0((x)))$ . Since  $K_0 = k_0(Z_0) \subset k_0((x))$  has no new  $\partial_x$ -constants it is a PPV-extension of  $k_0$  for Equation (3.3). Similarly  $K_1 = k_1(Z_0)$  is a PPV-extension of  $k_1$  for (3.3).

Let  $R_0 = k_0\{Z_0\}$  and  $R_1 = k_1\{Z_0\}$  (where  $\{\dots\}$  denotes the *differential ring* generated by  $\dots$ ). We may write  $R_0 = k_0\{Y\}/\mathcal{I}_0$  and  $R_1 = k_1\{Y\}/\mathcal{I}_1$  where  $Y$  is a set of  $n^2$  *differential* indeterminates and  $\mathcal{I}_0$  and  $\mathcal{I}_1$  are *differential* ideals in their respective rings. Exactly as in the non-parameterized case, one shows that  $\mathcal{I}_1 = k_1 \cdot \mathcal{I}_0$ .

For  $i = 0, 1$ , the PPV-group of  $K_i$  over  $k_i$  is the set of matrices  $B$  in  $\mathrm{GL}_n(C_i)$  such that  $Z_0B$  is again a zero of  $\mathcal{I}_i$ . Let  $\mathcal{I}_0 = \{f_j\}_{j \in J}$  for some indexing set  $J$ . We then have that the PPV-group of  $K_0$  over  $k_0$  is the set of  $B \in \mathrm{GL}_n(C_0)$  such that  $f_j(Z_0B) = 0$  for  $j \in J$ . We may consider  $f_j(Z_0B)$  as an element of  $C_0((x))$  and write  $f_j(Z_0B) = \sum_{\ell \geq \ell_0} f_{\ell,j}(B)x^\ell$  where  $f_{\ell,j}(B)$  is a *differential* polynomial in the entries of  $B$  with coefficient *in*  $C_0$ . Therefore the PPV-group of  $K_0$  over  $k_0$  is defined as

$$G_0 = \{B \in \mathrm{GL}_n(C_0) \mid f_{\ell,j}(B) = 0, \text{ for } j \in J \text{ and } \ell \geq \ell_0, \}.$$

Since  $\mathcal{I}_1 = k_1 \cdot \mathcal{I}_0$ , the differential ideal  $\mathcal{I}_1$  is generated by  $\{f_j\}_{j \in J}$  in  $k_1\{Y\}$ , and hence the PPV-group of  $K_1$  over  $k_1$  is defined as

$$G_1 = \{B \in \mathrm{GL}_n(C_1) \mid f_{\ell,j}(B) = 0, \text{ for } j \in J \text{ and } \ell \geq \ell_0, \}.$$

This means that if  $G$  is the linear differential algebraic group defined over  $C_0$  by  $\{f_{\ell,j}(B) = 0\}$ , then the PPV-group of  $K_i$  over  $k_i$  is  $G(C_i)$  for  $i = 0, 1$ , and since  $G$  is defined over  $C_0$  by the group of its  $C_0$ -points, this ends the proof.  $\square$

We shall now show that the “parameterized” monodromy matrices of Equation (3.2) are elements of the PPV-group.

Let  $\mathcal{D}$  be an open subset of  $\mathbb{P}^1(\mathbb{C})$  with  $0 \in \mathcal{D}$ . Assume that  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is the union of  $m$  disjoint disks  $D_i$  and that for each  $t \in \mathcal{U}$ , Equation (3.2) has a unique singular point in each  $D_i$ . Let  $\gamma_i$ ,  $i = 1, \dots, m$  be the obvious loops generating  $\pi_1(\mathcal{D}, 0)$ . Let us fix a fundamental solution of (3.2) in the neighborhood of 0 and define, for each fixed  $t \in \mathcal{U}$ , the monodromy matrices of (3.2) with respect to this solution and the  $\gamma_i$ . We will call these matrices, which depend on  $t$ , the *parameterized monodromy matrices* of Equation (3.2).

In the classical, non-parameterized situation, we gave an argument using analytic continuation of  $P(Z_0)$  where  $P$  had coefficients in  $C_0(x)$ . This argument made sense because the coefficients of  $P$  were *a fortiori* analytic functions. In the present situation, the coefficients, *a priori*, do not have such a meaning. The following result of Seidenberg [30, 31] allows us to give them such a meaning.

**Theorem 3.4** (Seidenberg). Let  $\mathbb{Q} \subset \mathcal{K} \subset \mathcal{K}_1$  be finitely generated differential extensions of the field of rational numbers  $\mathbb{Q}$ , and assume that  $\mathcal{K}$  consists of meromorphic functions on some domain  $\Omega \in \mathbb{C}^r$ . Then  $\mathcal{K}_1$  is isomorphic to a field  $\mathcal{K}_1^*$  of functions that are meromorphic on a domain  $\Omega_1 \subset \Omega$ , such that  $\mathcal{K}|_{\Omega_1} \subset \mathcal{K}_1^*$ .

Note that finitely generated here means finitely generated in the differential sense. We are now able to prove the same result as in the PV-case.

**Theorem 3.5.** Assume in Equation (3.3) that  $A \in \mathfrak{gl}_n(C_0(x))$ , where  $C_0$  is any differentially closed  $\Delta_t$ -field containing  $\mathbb{C}$  and let  $C_1$  be any differentially closed  $\Delta_t$ -field containing  $C_0$  and the entries of the parameterized monodromy matrices of Equation (3.2) with respect to a fundamental solution of (3.2). Then the parameterized monodromy matrices belong to  $G(C_1)$ , where  $G$  is the PPV-group of (3.2) over the  $\Delta$ -field  $C_0(x)$ .

*Proof.* As in the proof of Proposition 3.3, let us construct a formal fundamental solution  $Z_0 \in \text{GL}_n(C_0((x)))$  of Equation (3.3). The coefficients of the power series  $Z_0$  are matrices whose entries are polynomials, with integer coefficients, in the  $x$ -coefficients of the entries of  $A(x, t)$ , assuming as we are that  $Z_0$  has the initial condition  $Z_0(t, 0) = I_n$ . The coefficients of  $Z_0$  are therefore analytic on some domain  $\mathcal{U} \subset \mathbb{C}^r$ . The usual estimates show that  $Z_0$  is analytic in  $\mathcal{V} \times \mathcal{U}$  where  $\mathcal{V}$  is a neighborhood of 0 in the  $x$ -plane. Let  $\mathcal{K}$  be the differential  $\Delta_t$ -field generated over  $\mathbb{Q}$  by the  $x$ -coefficients of the entries of  $A(x, t)$  (that is, the coefficients of powers of  $x$  of these rational functions of  $x$ ) and the coefficients of the power series  $Z_0$ . Note that  $\mathcal{K}$  consists of functions meromorphic on  $\mathcal{U}$ . Let  $B$  be a parameterized monodromy matrix corresponding to some  $[\gamma]$ . For any differential polynomial  $P \in C_0(x)\{Y\}$  let us show that  $P(BZ_0) = 0$  whenever  $P(Z_0) = 0$ . As in the PV-case, this will prove that  $B \in G(C_0)$ . Let  $\mathcal{K}_1$  be the  $\Delta_t$ -field generated over  $\mathcal{K}$  by the entries of  $B$ . The Seidenberg theorem applied to  $\mathcal{K}$  and  $\mathcal{K}_1$  allows us to identify  $\mathcal{K}_1$  with a differential field of meromorphic functions on some  $\mathcal{U}_1 \subset \mathcal{U}$  and in particular to consider  $P(BZ_0)$  as a function analytic on  $\mathcal{V} \times \mathcal{U}_2$  for some  $\mathcal{U}_2 \subset \mathcal{U}_1$ . Let us continue  $P(Z_0)$  along  $\gamma$ , thought for each fixed  $t \in \mathcal{U}_2$  as a path in  $\mathcal{D} \times \{t\} \subset \mathcal{D} \times \mathcal{U}_2$ . Since the coefficients of  $P$  remain unchanged,  $P(BZ_0) = 0$  holds after continuation along  $\gamma$ .  $\square$

#### 4. A PARAMETERIZED ANALOGUE OF THE SCHLESINGER THEOREM

We consider as before a parameterized equation

$$(4.1) \quad \frac{\partial Y}{\partial x} = A(x, t)Y$$

where  $A(x, t)$  is a rational function of  $x$  with coefficients that are functions of a multi-variable  $t = (t_1, \dots, t_r)$ , analytic in some domain  $\mathcal{U} \in \mathbb{C}^r$  containing 0. We shall denote differentiation with respect to  $x$  and  $t_1, \dots, t_r$  by  $\partial_x$  and  $\partial_{t_1}, \dots, \partial_{t_r}$  respectively and let as before  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$ ,  $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$ .

The following lemma is inspired by a result of R. Palais ([26]).

**Lemma 4.1.** Let  $\mathcal{F}$  be a  $\Delta$ -field of functions that are meromorphic on  $\mathcal{V} \times \mathcal{U}$  where  $\mathcal{V} \subset \mathbb{C}$  and  $\mathcal{U} \subset \mathbb{C}^r$  are open connected sets, and let  $\mathcal{C}_x = \{u \in \mathcal{F} \mid \partial_x u = 0\}$ . Furthermore assume  $x \in \mathcal{F}$ . Let  $f \in \mathcal{F}$  be such that  $f(x, t) \in \mathbb{C}(x)$  for each  $t \in \mathcal{U}$ .

Then for some  $m \in \mathbb{N}$ , there exist  $a_0, \dots, a_m, b_0, \dots, b_m \in \mathcal{C}_x$  such that

$$f(x, t) = \frac{\sum_{i=0}^m a_i x^i}{\sum_{i=0}^m b_i x^i}$$

*Proof.* Let  $\mathcal{G} = \mathcal{F}(\mathbb{C})$  be the  $\Delta$ -field extension of  $\mathcal{F}$  generated by  $\mathbb{C}$ . For each  $r \in \mathbb{N}$ , let

$$X_r = \{t \in \mathcal{U} \mid f(x, t) \text{ is a ratio of polynomials of degrees at most } r\}.$$

By assumption  $\mathcal{U} = \cup_{r \in \mathbb{N}} X_r$ . The Baire Category Theorem implies that for some  $m$ , the closure of  $X_m$  has a nonempty interior. For  $t \in X_m$ , there exist  $a_{0,t}, \dots, a_{m,t}, b_{0,t}, \dots, b_{m,t} \in \mathbb{C}$  such that

$$f(x, t) = \frac{\sum_{i=0}^m a_{i,t} x^i}{\sum_{i=0}^m b_{i,t} x^i}.$$

This implies that, for  $t \in X_m$ ,

$$x^m f(x, t), x^{m-1} f(x, t), \dots, f(x, t), x^m, \dots, 1$$

are linearly dependent over  $\mathbb{C}$ . In particular the Wronskian determinant  $W(x, t) = \det(x^m f(x, t), x^{m-1} f(x, t), \dots, f(x, t), x^m, \dots, 1)$  (with respect to  $\partial_x$ ) vanishes for any  $t \in X_m$ . Since the closure of  $X_m$  has a nonempty interior,  $W(x, t) = 0$  on a nonempty open set and so is identically zero. Since

$$x^m f(x, t), x^{m-1} f(x, t), \dots, f(x, t), x^m, \dots, 1 \in \mathcal{F}$$

the vanishing of the Wronskian determinant implies that these elements are also linearly dependent over the  $\partial_x$ -constants of  $\mathcal{F}$ , that is,  $\mathcal{C}_x$ . Therefore there exist  $a_0, \dots, a_m, b_0, \dots, b_m \in \mathcal{C}_x$ , not all zero, such that  $(\sum_{i=0}^m b_i x^i) f(x, t) = \sum_{i=0}^m a_i x^i$ .  $\square$

This lemma is the key to prove an analogue of the Schlesinger theorem.

Consider Equation (4.1) and as in the previous section, let  $\mathcal{D}$  be an open subset of  $\mathbb{P}^1(\mathbb{C})$  with  $0 \in \mathcal{D}$ , assuming that  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is the union of  $m$  disjoint disks  $D_i$  and that for each  $t \in \mathcal{U}$ , Equation (4.1) has a unique singular point  $x_i(t)$  in each  $D_i$ . Let  $\gamma_i, i = 1, \dots, m$  be the obvious loops generating  $\pi_1(\mathcal{D}, 0)$ . We also assume that  $0 \in \mathcal{U}$ . We construct as before a fundamental solution  $Z_0$  of (4.1) near  $0 \in \mathcal{D}$ , analytic on  $\mathcal{V} \times \mathcal{U}$  where  $\mathcal{V}$  is some neighborhood of 0 in  $\mathcal{D}$ , and such that  $Z_0(t, 0) = I_n$ . By the theorem about analytic dependence on initial conditions (*cf.* [7]), the analytic continuation of  $Z_0$  along each  $\gamma_i$  (for each fixed  $t \in \mathcal{U}$ ) provides a solution which is again analytic on  $\mathcal{V} \times \mathcal{U}$ . We may therefore assume, on a possibly smaller  $\mathcal{U}$ , that  $B_{\gamma_i} \in \text{GL}_n(\mathcal{O}_{\mathcal{U}})$  for each parameterized monodromy matrix  $B_{\gamma_i}$  with respect to  $Z_0$ .

In what follows  $k(x)$ , for any differential  $\Delta_t$ -field  $k$ , will denote the field of rational functions in the indeterminate  $x$  and coefficients in  $k$ , where  $x$  is a  $\Delta_t$ -constant and  $\partial_x$  is the usual, formal differentiation of rational functions with  $\partial_x(x) = 1$  and  $\partial_x(a) = 0$  for all  $a \in k$ .

**Theorem 4.2.** With notation as before, assume that Equation (4.1) has regular singularities near each  $x_i(0)$ ,  $i = 1, \dots, m$ . Let  $k$  be a differentially closed  $\Delta_t$ -field containing the  $x$ -coefficients of the entries of  $A$ , the singularities  $x_i(t)$  of (4.1) and the entries of the parameterized monodromy matrices with respect to  $Z_0$ . Then the parameterized monodromy matrices generate a Kolchin-dense subgroup of  $G(k)$ , where  $G$  is the PPV-group of (4.1) over  $k(x)$ .

*Proof.* Under the assumptions of the theorem,  $K = k(x)\langle Z_0 \rangle$  is a PPV-extension of  $K$  for (4.1) (it has no new constants) and by Theorem 3.5 we know that the parameterized monodromy matrices lie in its PPV-group  $G(k)$ . We now wish to show that these generate a Kolchin-dense subgroup. Using the parameterized Galois correspondence it is enough to show, for any  $f \in K$ , that  $f \in k(x)$  whenever  $f$  is left invariant by the action of the monodromy matrices. Let  $\mathcal{F}_0$  be the differential  $\Delta_t$ -subfield of  $k$  generated over  $\mathbb{Q}$  by the  $x$ -coefficients of the entries of  $A$ , the singularities  $x_i(t)$  and the entries of the parameterized monodromy matrices with respect to  $Z_0$ . Note that  $\mathcal{F}_0$  consists of functions meromorphic on  $\mathcal{U}$ . Let us fix  $f \in K$ , and any finitely generated differential  $\Delta_t$ -subfield  $\mathcal{F}_1$  of  $k$  containing  $\mathcal{F}_0$ , such that  $f \in \mathcal{F}_1(x)\langle Z_0 \rangle$ . Using Theorem 3.4 we may consider  $\mathcal{F}_1$  as a field of meromorphic functions on some (possibly smaller than  $\mathcal{V} \times \mathcal{U}$ ) domain  $\Omega$  of the  $(x, t)$ -space. We may moreover assume that  $f$  is a rational function, with coefficients that are analytic functions on  $\Omega$  (rational in  $x$ , analytic in  $t$ ) in the entries of both  $Z_0$  and some  $\Delta_t$ -derivatives of  $Z_0$ . Assume that  $f$  is left invariant by the parameterized monodromy matrices, that is, by analytic continuation of  $f$  along each  $\gamma_i$ , for any fixed value of  $t$  such that  $(0, t) \in \Omega$ . For such fixed values of  $t$ , Corollary 2.6, 2) implies that  $f$  has moderate growth at its singular points. Therefore, for these fixed values of  $t$ ,  $f$  is a rational function of  $x$  with coefficients in  $\mathbb{C}$ . Lemma 4.1 implies that  $f$  is a rational function of  $x$  with coefficients in the subfield  $\mathcal{C}_x$  of  $\partial_x$ -constants of  $\mathcal{F}_1(x)\langle Z_0 \rangle$ , and since  $\mathcal{C}_x \subset k$ , this ends the proof.  $\square$

## 5. A WEAK PARAMETERIZED RIEMANN-HILBERT PROBLEM

Classically, the weak form of the Riemann-Hilbert Problem is:

*Let  $S = \{a_1, \dots, a_s\}$  be a finite subset of  $\mathbb{P}^1(\mathbb{C})$  and  $a_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ . Given a representation*

$$\rho : \pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

*of the fundamental group  $\pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$  show that there exists a linear differential system*

$$\frac{dY}{dx} = AY$$

*with  $A \in \mathfrak{gl}_n(\mathbb{C}(x))$ , having only regular singular points, all in  $S$ , such that for some fundamental solution analytic at  $a_0$ , the monodromy representation is  $\rho$ .*

Solutions of this problem go back to Plemelj with modern versions presented by Röhrl, Deligne and others (see [2] or [27] for presentations of solutions and historical references). In this section, we will present a solution to a parameterized version of this problem and apply this to the inverse problem in parameterized Picard-Vessiot Theory.

To state a parameterized version of the weak Riemann-Hilbert Problem, note that a representation of  $\pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$  above is determined by the images of the generators of this group, that is, by selecting  $s$  invertible matrices  $M_1, \dots, M_s$  such that  $M_1 \cdot \dots \cdot M_s = I_n$ , the identity matrix. We will present a solution of the following weak parameterized Riemann-Hilbert Problem

Let  $S = \{a_1, \dots, a_s\}$  be a finite subset of  $\mathbb{P}^1(\mathbb{C})$  and  $a_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ , and let  $D$  be an open polydisk<sup>1</sup> in  $\mathbb{C}^r$ . Let  $\gamma_1, \dots, \gamma_s$  be generators of  $\pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$  and, for  $i = 1, \dots, s$ , let  $M_i : D \rightarrow \mathrm{GL}_n(\mathbb{C})$  be analytic maps with  $M_1 \cdot \dots \cdot M_s = I_n$ . Show that there exists a parameterized linear differential system

$$\frac{\partial Y}{\partial x} = A(x, t)Y$$

with  $A \in \mathfrak{gl}_n(\mathcal{O}_{D'}(x))$  for some open polydisk  $D' \subset D$ , having only regular singular points, all in  $S$ , and such that for some fundamental solution, the parameterized monodromy matrix along each  $\gamma_i$  is  $M_i$ .

Let  $K$  be the field of functions, meromorphic on  $D$ , generated over  $\mathbb{Q}$  by the entries of the matrices  $M_i$ . The assignment  $\gamma_i \mapsto M_i$  yields a homomorphism  $\chi : \pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0) \rightarrow \mathrm{GL}_n(K)$ .

Modern solutions of the weak Riemann-Hilbert Problem use the techniques of analytic vector bundles and we will proceed in a similar fashion following the presentation in [6]. We begin by first constructing a family of vector bundles on  $\mathbb{P}^1(\mathbb{C})$  together with a family of meromorphic connections. We will then show how this yields a solution of the weak parameterized Riemann-Hilbert Problem. We note that the related but different problem of constructing an *isomonodromic* family including a given Fuchsian differential equation is considered in [23] and [21] (see also [16] and [28]).

Vector bundles are determined by cocycles with respect to coverings so we will proceed by defining parameterized cocycles on  $\mathbb{P}^1(\mathbb{C})$ .

We begin by considering  $\mathbb{P}^1(\mathbb{C}) \setminus S$  (and will fill in the ‘‘holes’’ later).

Let  $U_{s+1}, \dots, U_N$  be a covering of  $\mathbb{P}^1(\mathbb{C}) \setminus S$  by open disks (other disks  $U_1, \dots, U_s$  will be defined below) and let  $\eta_i$  be a path in  $\mathbb{P}^1(\mathbb{C}) \setminus S$  from  $x_0$  to the center of  $U_i$ . For each pair  $(i, j)$  with  $i < j$  and  $U_i \cap U_j \neq \emptyset$ , let  $\delta_{i,j}$  be the line-segment joining the center of  $U_i$  to the center of  $U_j$ . Let

$$g_{i,j} = \chi([\eta_i \circ \delta_{i,j} \circ \eta_j^{-1}]) \in \mathrm{GL}_n(K).$$

By replacing  $D$  with a smaller open polydisk if necessary, we may assume that the entries of all the  $g_{i,j}$  are analytic functions on  $D$ . For each  $t \in D$ , we consider  $g_{i,j}(t) : U_i \cap U_j \rightarrow \mathrm{GL}_n(\mathbb{C})$  as a constant function on  $U_i \cap U_j$ . One can easily check that, for fixed  $t \in D$ , the  $\{g_{i,j}(t)\}$  form a cocycle and therefore define an analytic vector bundle  $\hat{\mathcal{F}}(t)$  of rank  $n$  over  $\mathbb{P}^1(\mathbb{C}) \setminus S$ . For later use it is important to note that the  $\{g_{i,j}\}$  can also be thought of as forming a cocycle on  $(\mathbb{P}^1(\mathbb{C}) \setminus S) \times D$  with respect to the covering  $\{U_i \times D\}$  and so define an analytic vector bundle  $\hat{\mathcal{F}}$  on  $(\mathbb{P}^1(\mathbb{C}) \setminus S) \times D$ .

For each  $t \in D$ , we define a system of linear differential equations locally over each  $U_i, i = s+1, \dots, N$  via the forms

$$dy = \omega_i(t)y$$

---

<sup>1</sup>An open polydisk  $D$  is a set of the form  $\{t = (t_1, \dots, t_r) \mid |t_j - w_j| < r_j \ 1 \leq j \leq r\}$  for some  $w = (w_1, \dots, w_r)$  and positive real numbers  $r_j$ .

where each  $\omega_i(t) = 0$ . Of course, this is just the trivial system and one has, on each nonempty intersection  $U_i \cap U_j$ ,

$$\omega_i(t) = d(g_{i,j}(t))(g_{i,j}(t))^{-1} + (g_{i,j}(t))\omega_j(t)(g_{i,j}(t))^{-1}$$

since the  $g_{i,j}(t)$  are constant on these sets. Therefore these local systems patch together to form an analytic connection  $\hat{\nabla}(t)$  on  $\hat{\mathcal{F}}(t)$ . Trivially, if we write  $\hat{\nabla}(t)$  in terms of local coordinates, one has that the terms appearing depend analytically on  $t$  (and this holds for any other trivializing covering as well). Furthermore, for each  $t \in D$ , the monodromy associated with simple loops  $\gamma_i$  from  $x_0$  surrounding each  $x_i$  corresponds to  $M_i(t)$ .

We will now fill in the ‘‘holes’’ and extend each  $(\hat{\mathcal{F}}(t), \hat{\nabla}(t))$  to a holomorphic vector bundle  $\mathcal{F}(t)$  on all of  $\mathbb{P}^1(\mathbb{C})$  and meromorphic connection  $\nabla(t)$  on  $\mathcal{F}(t)$ . Let  $U_1, \dots, U_s$  be pairwise disjoint open disks centered at  $a_1, \dots, a_s$  (for simplicity, we will assume that these are all finite points). Shrinking  $D$  again if necessary (but keeping the same notation), there exists for each  $i$  an analytic function  $N_i(t) : D \rightarrow \text{gl}_n(\mathbb{C})$  such that  $N_i(t) = (1/2\pi i) \log(M_i(t))$ . On each  $U_i$ ,  $1 \leq i \leq s$ , consider for fixed  $t \in D$  the meromorphic system

$$(5.1) \quad dy = \frac{N_i(t)}{(x - a_i)} d(x - a_i) y.$$

For each  $i$ ,  $1 \leq i \leq s$ , select an  $\alpha$  such that  $U_i \cap U_\alpha \neq \emptyset$  and let  $Y_i(x, t)$  be a solution of (5.1) on this latter set. We may write

$$Y_i(x, t) = (x - a_i)^{N_i(t)}.$$

on  $U_i \cap U_\alpha$ . Let  $g_{i,\alpha} = Y_i(x, t)$ . If  $U_\beta, \beta \neq \alpha$ , also has nonempty intersection with  $U_i$ , we consider a path in  $U_i$  from a designated point  $u_i \in U_i \cap U_\alpha$  ending in  $U_i \cap U_\beta$  moving in the counterclockwise direction (less than one turn) around  $a_i$ . We let  $g_{i,\beta}$  denote the analytic continuation of  $g_{i,\alpha}$  along this path. One can show that the  $g_{k,l}$  thus defined for all  $1 \leq k, l \leq N$  such that  $U_k \cap U_l \neq \emptyset$ , define a cocycle for the covering  $\{U_l \times D\}_{1 \leq l \leq N}$  of  $\mathbb{P}^1(\mathbb{C}) \times D$ . This cocycle yields an analytic vector bundle  $\mathcal{F}$  on  $\mathbb{P}^1(\mathbb{C}) \times D$ . *A fortiori*, for each  $t \in D$ ,  $\{g_{k,l}(t)\}$  defines a cocycle for the covering  $\{U_l\}_{1 \leq l \leq N}$  of  $\mathbb{P}^1(\mathbb{C})$  and this yields, for each  $t \in D$  an analytic vector bundle  $\mathcal{F}(t)$  on  $\mathbb{P}^1(\mathbb{C})$ .

We now claim that for each  $t \in D$  the local systems  $\{dy = \omega_i(t)y\}_{i=1}^N$  define a meromorphic connection on  $\mathbb{P}^1(\mathbb{C})$ . We need only check compatibility on sets of the form  $U_i \cap U_\alpha$  with  $1 \leq i \leq s, s+1 \leq \alpha \leq N$ . On such a set we have

$$d(g_{i,\alpha}(t))g_{i,\alpha}(t)^{-1} + (g_{i,\alpha}(t))\omega_\alpha(t)(g_{i,\alpha}(t))^{-1} = dY_i Y_i^{-1} = \omega_i(t)$$

which proves compatibility. Therefore, for each  $t \in D$ , we have a connection  $\nabla(t)$  on  $\mathcal{F}(t)$ . Note that in local coordinate, the terms of  $\nabla(t)$  depend analytically on  $t$ . We refer to the pair  $(\mathcal{F}(t), \nabla(t))$  as the *canonical extension* of  $(\hat{\mathcal{F}}(t), \hat{\nabla}(t))$ , and this generalizes similar notions in the non-parameterized case introduced by Deligne. Note that the connection  $\nabla(t)$  has at worst logarithmic poles at the  $a_i, i = 1, \dots, s$ . Therefore, the differential systems will have regular (even Fuchsian) singular points in the local coordinates.

We now wish to change the covering of  $\mathbb{P}^1(\mathbb{C}) \times D$  and consider the vector bundle  $\mathcal{F}$  with respect to this new covering. In particular, let  $p_1 \neq p_2$  be points of  $\mathbb{P}^1(\mathbb{C})$

and let  $V_1 = \mathbb{P}^1(\mathbb{C}) \setminus \{p_2\}$  and  $V_2 = \mathbb{P}^1(\mathbb{C}) \setminus \{p_1\}$ . We claim that  $\mathcal{F}$  is isomorphic to a vector bundle determined by a single cocycle  $\bar{g}_{1,2} : (V_1 \cap V_2) \times D \rightarrow \mathrm{GL}_n(\mathbb{C})$ . First note that both  $V_1 \times D$  and  $V_2 \times D$  are contractible topological spaces so any vector bundle over these spaces is topologically trivial. Furthermore, both of these sets are Stein manifolds ([15], p. 209) so any topologically trivial analytic vector bundle is analytically trivial ([14], Satz 2, [12], Corollary 3.2). Therefore,  $V_1 \times D$  and  $V_2 \times D$  form a cover such that the vector bundle is analytically trivial on each set. This implies that the vector bundle is indeed determined by a cocycle  $\bar{g}_{1,2} : (V_1 \cap V_2) \times D \rightarrow \mathrm{GL}_n(\mathbb{C})$ .

Under the isomorphism described above, the connections  $\nabla(t)$  have a new coordinate description. If we consider the covering  $\{U_i\}_{1 \leq i \leq N}$  of  $\mathbb{P}^1(\mathbb{C})$ , the isomorphism yields equivalent cocycles  $\bar{g}_{i,j}(t) = \Gamma_i(t)^{-1} g_{i,j}(t) \Gamma_j(t)$  where  $\Gamma_i(t)$  and  $\Gamma_j(t)$  are analytically invertible matrices on their respective coordinate patches. Since we are dealing with an isomorphism of vector bundles over  $\mathbb{P}^1(\mathbb{C}) \times D$ , one sees that the  $\Gamma_i(t)$  are holomorphic in  $t$  as well. The local forms of the connection in this new coordinate description become

$$\begin{aligned} dy &= \bar{\omega}_i(t)y, \text{ where} \\ \bar{\omega}_i(t) &= d(\Gamma_i(t))\Gamma_i(t)^{-1} + \Gamma_i(t)\omega_i(t)\Gamma_i(t)^{-1}. \end{aligned}$$

Note that the  $\bar{\omega}_i(t)$  also has, at worst, poles of order 1 at the  $a_j$ .

We now turn to the solution of the weak parameterized Riemann-Hilbert Problem. In the solution of the Riemann-Hilbert Problem as presented in Chapter 3 of [2], the authors appeal to the Birkhoff-Grothendieck Theorem. To prove the existence of a system (5.2) that solves the parameterized weak Riemann-Hilbert Problem, we shall use a parameterized version of the classical Birkhoff-Grothendieck Theorem. We can assume that  $s \geq 2$  (otherwise the Riemann-Hilbert Problem is trivial). Consider the vector bundle  $\mathcal{F}$  on  $\mathbb{P}^1(\mathbb{C}) \times D$  with respect to the covering  $\{V_1 \times D, V_2 \times D\}$ , as defined above, where  $p_1 = a_1$  and  $p_2 = a_2$ . For simplicity of notation we will assume that  $a_1 = 0$  and  $a_2 = \infty$ . Let  $\bar{g}_{1,2} : (V_1 \cap V_2) \times D \rightarrow \mathrm{GL}_n(\mathbb{C})$  be the associated cocycle. The parameterized Birkhoff-Grothendieck Theorem (*cf.* [21], Proposition 4.1; [4], Theorem 2; [5], Theorem A.1) states:

*There exists an open polydisk  $D' \subset D$  as well as maps  $\Phi_1 : V_1 \times D' \rightarrow \mathrm{GL}_n(\mathbb{C})$  with  $\Phi_1, \Phi_1^{-1}$  analytic on  $V_1 \times D'$ , and  $\Phi_2 : V_2 \times D' \rightarrow \mathrm{GL}_n(\mathbb{C})$ , with  $\Phi_2, \Phi_2^{-1}$  analytic on  $V_2 \times D'$  such that*

$$\bar{g}_{1,2} = \Phi_1^{-1} x^\Lambda \Phi_2$$

*where  $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$  for integers  $\lambda_1 \geq \dots \geq \lambda_n$ .*<sup>2</sup>

We will now consider (an isomorphic copy of)  $\mathcal{F}$  determined by  $\Phi_1$  and  $\Phi_2$ . Note that the cocycle defining this vector bundle is  $(x - a_1)^\Lambda : V_1 \cap V_2 \times D' \rightarrow \mathrm{GL}_n(\mathbb{C})$ . In these new coordinates, for each  $t \in D'$ , the connection  $\nabla(t)$  above  $V_1$  corresponds to a linear differential equation

$$\frac{\partial Y}{\partial x} = A_1(x, t)Y$$

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<sup>2</sup>One may also select  $D'$  to be  $D \setminus \Sigma$  where  $\Sigma$  is an analytic subset of codimension 1 and assert that  $\Phi_1$  and  $\Phi_2$  are meromorphic along  $\Sigma$  but we shall not need this stronger version.

where  $A_1$  is analytic in  $t$  and analytic in  $x$  outside of  $\{a_1, a_3, \dots, a_s\}$  and having poles of order at most one at these points. Above  $V_2$  we have the expression

$$\frac{\partial Y}{\partial u} = A_2(u, t)Y$$

where  $A_2$  is analytic in  $t$  and analytic in  $u = 1/x$  outside of  $x \in \{a_2, a_3, \dots, a_s\}$  and having poles of order at most one at these points. Using the cocycle, we have

$$A_1(x, t) = \frac{\partial(x^\Lambda)}{\partial x}x^{-\Lambda} + x^\Lambda A_2(1/x, t)x^{-\Lambda}.$$

Since the right-hand side of this equation represents a function meromorphic at  $x = \infty$ , we have that  $A_1$  is meromorphic at  $\infty$  as well. Lemma 4.1 implies that the entries of  $A_1(x, t)$  are rational in  $x$  with coefficients that are functions analytic on  $D'$ . We therefore have the following solution of the weak parameterized Riemann-Hilbert Problem.

**Theorem 5.1.** Let  $S = \{a_1, \dots, a_s\}$  be a finite subset of  $\mathbb{P}^1(\mathbb{C})$  and  $D$  an open polydisk in  $\mathbb{C}^r$ . Let  $\gamma_1, \dots, \gamma_s$  be generators of  $\pi^1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$  for some fixed base-point  $a_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$ , and let  $M_i : D \rightarrow \mathrm{GL}_n(\mathbb{C})$ ,  $i = 1, \dots, s$ , be analytic maps with  $M_1 \cdot \dots \cdot M_s = I_n$ . There exists a parameterized linear differential system

$$\frac{\partial Y}{\partial x} = A(x, t)Y$$

with  $A \in \mathfrak{gl}_n(\mathcal{O}_{D'}(x))$  for some open polydisk  $D' \subset D$ , having only regular singular points, all in  $S$ , such that for some parameterized fundamental solution, the parameterized monodromy matrix along each  $\gamma_i$  is  $M_i$ . Furthermore, given any  $a_i \in \{a_1, \dots, a_s\}$ , the entries of  $A$  can be chosen to have no or simple poles at all  $a_j \neq a_i$ .

As an application of Theorem 4.2 and Theorem 5.1, we solve the inverse problem of parameterized Picard-Vessiot Theory in a special case. Let  $k$  be a so-called  $\Delta_t$ -universal field, that is, a  $\Delta_t$ -field such that for any  $\Delta_t$ -field  $k_0 \subset k$ ,  $\Delta_t$ -finitely generated over  $\mathbb{Q}$  and any  $\Delta_t$ -finitely generated extension  $k_1$  of  $k_0$ , there is a  $\Delta_t$ -differential  $k_0$ -isomorphism of  $k_1$  into  $k$  (see [19], Chapter III, §7). Note that  $k$  is in particular a differentially closed  $\Delta_t$ -field. As in Section 4, let  $k(x)$  denote the  $\Delta = \{\partial_x, \partial_{t_1}, \dots, \partial_{t_r}\}$ -field of rational functions in the indeterminate  $x$  with coefficients in  $k$ , where  $x$  is a  $\Delta_t$ -constant with  $\partial_x(x) = 1$  and  $\partial_x$  commutes with the  $\partial_{t_i}$ . We shall prove the following

**Corollary 5.2.** Let  $G$  be a  $\Delta_t$ -linear differential algebraic group defined over  $k$  and assume that  $G(k)$  contains a finitely generated subgroup  $H$  that is Kolchin-dense in  $G(k)$ . Then  $G(k)$  is the PPV-group of a PPV-extension of  $k(x)$ .

*Proof.* Let  $M_1, \dots, M_s$  generate  $H$ . We may assume that  $M_1 \cdot \dots \cdot M_s = I_n$ . Let  $k_0$  be a finitely generated  $\Delta_t$ -field containing the entries of the  $M_i$ . Theorem 3.4 (with  $\mathcal{K} = \mathbb{Q}$  and  $\mathcal{K}_1 = k_0$ ) implies that there is an open polydisk  $D$  in  $\mathbb{C}^r$  such that we may consider the elements of  $k_0$  as meromorphic functions on  $D$ . By shrinking  $D$  if necessary, we may assume that the entries of the  $M_i$  are analytic on  $D$ . Theorem 5.1 implies that there exists a parameterized linear differential system

$$\frac{\partial Y}{\partial x} = A(x, t)Y$$

with  $A \in \mathfrak{gl}_n(\mathcal{O}_{D'}(x))$  for some open polydisk  $D' \subset D$ , having only regular singular points, all in  $S$ , such that for some fundamental solution, the parameterized monodromy matrix along each  $\gamma_i$  is  $M_i$ . Let  $k_1$  be a (differentially) finitely generated  $\Delta_t$ -extension of  $k_0$  containing the  $x$ -coefficients of the entries of  $A$ . Since  $k$  is  $\Delta_t$ -universal, we may assume that  $k_1 \subset k$ . Theorem 4.2 implies that the group generated by the  $M_i$  is Kolchin-dense in the PPV-extension corresponding to  $\frac{\partial Y}{\partial x} = A(x, t)Y$  and so this PPV-group must be  $G(k)$ .  $\square$

This proof follows closely the proof in [34] where the authors show than any linear algebraic group defined over  $\mathbb{C}$  is a Galois group of a Picard-Vessiot extension. Those authors use a solution of the weak Riemann-Hilbert Problem, Schlesinger's Theorem and the fact that any linear algebraic group contains a Zariski dense finitely generated subgroup. In contrast, not all linear differential algebraic groups contain dense finitely generated subgroups. For example, the Kolchin-closure of any finitely generated subgroup  $H$  of the additive group  $\mathbb{G}_a(k)$  is a proper subgroup of  $\mathbb{G}_a(k)$ . A proof of this in the ordinary case, that is, when  $\Delta_t = \{\partial_t\}$  proceeds as follows. Let  $H$  be generated by  $z_1, \dots, z_m$  and assume  $z_1, \dots, z_s$  is a basis for the  $C$ -vector space spanned by  $H$ . The elements of the group  $H$  all satisfy  $L(y) = 0$  for  $L(y) = wr(y, z_1, \dots, z_s)$  where  $wr(\dots)$  denotes the wronskian determinant. In [20] and [8] it is furthermore shown that  $\mathbb{G}_a(k)$  as well as  $\mathbb{G}_m(k)$  is not the PPV-group of any PPV-extension of  $k(x)$ . Among linear algebraic groups, this group presents the main obstruction to a linear algebraic group being a PPV-group over  $k(z)$  since it is shown in [32] that a linear algebraic group  $G$  defined over  $k$  is a PPV-group of a PPV-extension of  $k(x)$  if and only if the identity component of  $G$  has no quotient isomorphic to  $\mathbb{G}_a(k)$  or  $\mathbb{G}_m(k)$ . The proof of this latter fact relies on Proposition 5.2.

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INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ DE STRASBOURG ET CNRS,  
7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE  
*E-mail address:* mitschi@math.unistra.fr

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, BOX 8205, RALEIGH,  
NORTH CAROLINA 27695-8205  
*E-mail address:* singer@math.ncsu.edu