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SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN FUNCTION FIELDS OF ONE VARIABLE

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ABSTRACT. Formal power series techniques are used to investigate the algebraic relationships between a function satisfying a linear differential equation and its derivatives. We are able to derive some conclusions, among them that an elliptic function satisfies no linear differential equation over a liouvillian extension of the complex numbers.

In [3], Rosenlicht noticed that if an element y belonged to a liouvillian extension of a differential field, then the zeroes and poles of it and its derivatives must satisfy certain relations. His main tool was

THEOREM. *Let K be a field of characteristic zero, k a subfield of K, P a real discrete k-place of K whose residue field is algebraic over k, D a derivation of K that is continuous in the topology of P and that maps k into itself. Let x, y be nonzero elements of K such that each of $x(P)$, $y(P)$ is either* 0 *or* ∞. *Then*:

(1) *If* $\operatorname{ord}_P(Dx/x) \geqq 0$, *then* $\operatorname{ord}_P(Dy/y) \geqq 0$. *Here D induces a derivation on the residue field of P. Denoting this residue field derivation by the same symbol D, for any z in K such that* $\operatorname{ord}_P z \geqq 0$, *we have* $(Dz)(P) = D(z(P))$.

(2) *If* $\operatorname{ord}_P(Dx/x) < 0$, *then* $\operatorname{ord}_P(Dx/x) = \operatorname{ord}_P(Dy/y)$ *and, therefore,* $\operatorname{ord}_P(y/x) = \operatorname{ord}_P(Dy/Dx)$. *In addition,* $(y/x)(P) = (Dy/Dx)(P)$.

Using this fact, he was able to show that certain differential equations have no liouvillian solutions. In this paper, we will show that the poles and zeroes of a solution of a linear differential equation and its derivatives must satisfy certain relations. With this we are able to mimic Rosenlicht's results and show that solutions of a large class of differential equations satisfy no linear differential equation (Corollaries 1 and 2). We will also prove a strengthened version of results of C. L. Siegel [5, p. 60] and L. Goldman [1, Corollary 3] and give an easy proof of a structure theorem of L. Goldman [1, Corollary 4].

The main tool of this paper is

LEMMA. *Let $k \subset K$ be differential fields of characteristic* 0. *Let $w \in K$ satisfy the linear differential equation*

$$w^{(n)} - A_{n-1}w^{(n-1)} - \cdots - A_0 w = B$$

with the A_i, B in k. Let P be a discrete k-place of $k\langle w\rangle$ such that the derivation $'$ is continuous in the topology of this place. Then $\operatorname{ord}_P w < 0$ *implies that* $\operatorname{ord}_P(w'/w) \geqq 0$.

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PROOF. Assume not; then $\mathrm{ord}_P(w'/w) < 0$ and so $\mathrm{ord}_P w' < 0$. Case 2 of the theorem now applies. We can conclude that $\mathrm{ord}_P(w''/w') = \mathrm{ord}_P(w'/w) < 0$ and $\mathrm{ord}_P w'' < 0$. Similarly $\mathrm{ord}_P(w^{(k)}/w^{(k-1)}) < 0$ and $\mathrm{ord}_P w^{(k)} < 0$, and, in particular, $\mathrm{ord}_P(w^{(n)}/w^{(n-1)}) < 0$.Using our linear equation, we have

$$w^{(n)}/w^{(n-1)} = B/w^{(n-1)} + A_{n-1} + A_{n-2}w^{(n-2)}/w^{(n-1)} + \cdots + A_0 w/w^{(n-1)}.$$

I claim that the right-hand side of this equation has order 0, which would give us a contradiction, and thus prove the lemma. First note that since

$$\mathrm{ord}_P(w^{(n-2)}/w^{(n-1)}) > 0 \text{ and } \mathrm{ord}_P(w^{(n-3)}/w^{(n-2)}) > 0,$$

we have $\mathrm{ord}_P(w^{(n-3)}/w^{(n-1)}) > 0$. Continuing in this way, we see that $\mathrm{ord}_P(w^{(n-i)}/w^{(n-1)}) > 0$ for $2 \leqq i \leqq n-1$. Also since $\mathrm{ord}_P w^{(n-1)} < 0$, $\mathrm{ord}_P(B/w^{(n-1)}) > 0$. Therefore, the order of the right-hand side of the above equation is 0.

COROLLARY 1. *Let $k \subset K$ be differential fields of characteristic 0 and $y \in K$. Let f be a polynomial in several variables over k of total degree less than n, some positive integer, and $y^n = f(y, y', y'', \ldots)$. Assume further that the transcendence degree of $k\langle y\rangle$ over k is 1. Then y satisfies no linear differential equation with coefficients in k.*

PROOF. Note that the transcendence degree assumption allows us to assume that the derivation is continuous in the topology of every k-plane [4, Lemma 1]. Assume that y did satisfy such an equation. By the lemma, we would then have $\mathrm{ord}_P(y'/y) \geqq 0$, where P is a pole of y. This, in turn, implies that $\mathrm{ord}_P y^{(m)} \geqq \min(0, \mathrm{ord}_P y)$ for all m. Since $\mathrm{ord}_P y < 0$, we have

$$\mathrm{ord}_P f(y, y', y'', \ldots) \geqq (n-1)\mathrm{ord}_P y > n(\mathrm{ord}_P y)) = \mathrm{ord}_P y^n,$$

which is a contradiction.

COROLLARY 2. *An elliptic function satisfies no linear differential equation with coefficients in a liouvillian extension of the complex numbers.*

PROOF. Let k be a liouvillian extension of the complex numbers and y an elliptic function. Since y satisfies the differential equation $(y')^2 = y^3 + ay + b$, for some $a, b \in \mathbf{C}$, $a^3/27 + b^2/4 \neq 0$, we could apply Corollary 1, once we know that the transcendence degree of $k\langle y\rangle$ over k is 1. By looking at the above differential equation, we know it is at most 1. If it were less, then y would lie in a liouvillian extension of the complex numbers, contradicting the results on p. 372 of [3].

A homogeneous linear differential polynomial $L(W)$, with coefficients in k, is said to be linearly reducible over k if there exist homogeneous linear differential polynomials $M(W)$, $N(W)$, each of positive order, with coefficients in k, such that $L(W) = M(N(W))$. If $L(W)$ is not linearly reducible over k, it is said to be irreducible over k. We will need the following fact relating the reducibility of $L(W)$ to the behavior of its solutions under isomorphisms. Let U be a universal extension of k with constant field C [2, p. 133], and x a nonzero element of U such that $L(x) = 0$. Let S be the set of differential k-isomorphisms of $k\langle x\rangle$ into U, r the dimension of T, the C-span of

$\{\sigma x | \sigma \in S\}$ over C, and n the order of $L(W)$. I claim that there exist homogeneous linear differential polynomials $L_{n-r}(W)$ and $L_r(W)$, of order $n-r$ and r, with coefficients in k, such that $L(W) = L_{n-r}(L_r(W))$.

To see this, we can assume that r is less than n, and let $\sigma_1 x, \sigma_2 x, \ldots, \sigma_r x$ be a C-basis of T and $L_r(W) = \mathrm{Wr}(W, \sigma_1 x, \ldots, \sigma_r x)/\mathrm{Wr}(\sigma_1 x, \ldots, \sigma_r x)$, where $\mathrm{Wr}(y_1, \ldots, y_m)$ is the Wronskian determinant. Any isomorphism of $k\langle\sigma_1 x, \ldots, \sigma_r x\rangle$ into U sends each $\sigma_i x$ into T and so leaves the coefficients of $L_r(W)$ fixed. By the corollary on p. 388 of [2], the coefficients of $L_r(W)$ must be in k. Let $v_1 = \sigma_1 x, v_2 = \sigma_2 x, \ldots, v_r = \sigma_r x, v_{r+1}, \ldots, v_n$ be a fundamental system of solutions of $L(W)$ in U. Every differential k-isomorphism of $k\langle L_r(v_{r+1}), \ldots, L_r(v_n)\rangle$ into U sends each $L(v_{r+i})$ into the C-span of $L(v_{r+1}), \ldots, L(v_n)$ and so leaves the coefficients of

$$L_{n-r}(W) = \mathrm{Wr}(W, L(v_{r+1}), \ldots, L(v_n))/\mathrm{Wr}(L(v_{r+1}), \ldots, L(v_n))$$

fixed. Therefore, $L_{n-r}(W)$ also has its coefficients in k. Since the coefficient of $W^{(n)}$ in both $L(W)$ and $L_{n-r}(L_r(W))$ is 1, $L(W) - L_{n-r}(L_r(W))$ is a homogeneous linear differential polynomial of order less than n, with n linearly independent solutions. Therefore $L(W) = L_{n-r}(L_r(W))$. In particular, if $L(W)$ is irreducible it has a fundamental set of solutions of the form $x, \sigma_1 x, \ldots, \sigma_{n-1} x$, where x is any nonzero solution and the σ_i's are differential k-isomorphisms of $k\langle x\rangle$ into U.

COROLLARY 3. *Let $k \subset K$ be differential fields of characteristic* 0 *and $w \in K$ which satisfies the linear differential equation $L(W) = B$, where $L(W) = W^{(n)} - A_{n-1}W^{(n-1)} - \cdots - A_0 W$ and the A_i and B are in k. If the transcendence degree of $k\langle w\rangle$ over k equals* 1, *then the homogeneous equation $L(W) = 0$ has a solution u such that u'/u is algebraic over k. If $L(W)$ is irreducible over k, then $L(W) = 0$ has a fundamental set of solutions $u_1, \ldots, u_n$ such that each u_i'/u_i is algebraic over k.*

PROOF. The second assertion follows from the first and the remark at the end of the preceding paragraph.

To prove the first assertion, let P be a pole of w. By the lemma, we have $\mathrm{ord}_P(w'/w) \geqq 0$. Using case 1 of the Theorem, and observing that $w' = w'w/w$,

$$w'' = ((w'/w)' + (w'/w)^2)w,$$

$$w''' = ((w'/w)'' + 3(w'/w)'w'/w + (w'/w)^3)w, \ldots,$$

$$w^{(n)} = ((w'/w)^{(n-1)} + n(w'/w)^{n-2}w'/w + \cdots + (w'/w)^n)w,$$

we see that $(w'/w)(P)$ is an algebraic solution of the equation

$$V^{(n-1)} + nV^{(n-2)} + \cdots + V^n - A_1(V^{(n-2)} + \cdots + V^{n-1}) - \cdots - A_n = (B/w)(P) = 0.$$

We can now find a u in some differential extension field of $k((w'/w)(P))$ such that $u'/u = (w'/w)(P)$. This u will then satisfy the homogeneous linear differential equation $L(W) = 0$.

COROLLARY 4. *Let $k \subset K$ be differential fields of characteristic* 0 *and $z \in K$ a solution of a linear differential equation with coefficients in k. Assume that the transcendence degree of $k\langle z\rangle$ over k is less than or equal to* 1. *Letting $\bar{k}$ be the algebraic closure of k, we can then find a v in $\bar{k}\langle z\rangle$ such that z is algebraic over $k\langle v\rangle$ and v satisfies a linear differential equation of order* 1 *over $\bar{k}$.*

PROOF. If the transcendence degree of $k\langle z\rangle$ over k is zero, we are done. Assume z is transcendental over k. Let v be an element of $\bar{k}\langle z\rangle$, transcendental over k, which satisfies a linear differential equation over $\bar{k}$ of least order r, which we may assume is bigger than 1. Let $L(V) = B$ be a linear differential equation of order r that v satisfies. By Corollary 3 we know that $L(V) = 0$ has a solution u such that u'/u is in $\bar{k}$. Letting S be the set of differential $\bar{k}$ -isomorphisms of $\bar{k}\langle u\rangle$ into the universal domain U, the dimension of the C-span of $\{\sigma u | \sigma \in S\}$ is 1. Using the paragraph preceding Corollary 3, we can conclude that $L(V) = L_{r-1}(L_1(V))$, where $L_{r-1}(V)$ and $L_1(V)$ are homogeneous linear differential polynomials of order $r - 1$ and 1, with coefficients in $\bar{k}$. $L_1(v)$ is in $\bar{k}\langle z\rangle$ and satisfies $L_{r-1}(V) = B$, a linear differential equation of order less than r. Therefore, $L_1(v)$ must be in $\bar{k}$ and v satisfies a linear differential equation of order 1 over $\bar{k}$, a contradiction. Therefore, $r = 1$.

Both Corollary 4 and a weaker form of Corollary 3 were proven by L. Goldman [1] using the theory of differential polynomials and, in particular, the leading coefficient theorem of Ritt, which we have avoided.

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