JOURNAL OF DIFFERENTIAL EQUATIONS 75, 354–370 (1988)

Autonomous Functions

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Received September 26, 1987

By definition, an autonomous function is a differentially algebraic function f on $\mathbb{R}$ (or on $\mathbb{C}$), every translate f_τ of which satisfies every algebraic differential equation that f satisfies. We find several equivalent formulations of the property of being autonomous. Our main result is that if f is differentially algebraic and meromorphic in the full complex plane, and if g is an autonomous entire function, then $f \circ g$ must be autonomous.

1. INTRODUCTION

A function $y(x)$ is said to be differentially algebraic (DA) if it satisfies a non-trivial algebraic differential equation (ADE), that is, an equation of the form

$$P(x, \mathbf{y}) = P(x, y(x), y'(x), ..., y^{(n)}(x)) = 0, \tag{1}$$

where P is a polynomial with complex coefficients in its $n+1$ variables—for example,

$$(x^2+2)\, y'''^3 y'^2 - 5(x+\pi)\, y''^2 y^3 + (7x+3) = 0.$$

Equation (1) is called *autonomous* if P is independent of the independent variable x—for example,

$$7y'''^4 y^2 + \sqrt{\pi}\, y''^3 y' - 3 = 0. \tag{2}$$

* The research of both authors was partially supported by grants from the National Science Foundation.

0022-0396/88 $3.00

(It is a simple fact (see [OST]) that every C^∞ function that is DA must satisfy an autonomous ADE.)

Throughout this paper, we restrict ourselves to functions that are meromorphic in the whole complex plane unless it is clear from the context otherwise. It is an amusing exercise to prove that e^x, or indeed any periodic meromorphic DA function $y(x)$, has the property that if $y(x)$ satisfies an ADE $P=0$, then every translate $y_\tau(x)=y(x-\tau)$ must also satisfy $P=0$. We shall call a function with this property *autonomous*, and study such functions here. An autonomous function is a DA function that cannot be distinguished from its translates by means of differential algebra. By the Shannon–Pour-El–Lipshitz–Rubel theorem (see [LIR]), this means (in the analytic case, at least) that if $y(x)$ is autonomous, then every general-purpose analog computer that produces $y(x)$ also produces every translate of $y(x)$.

In Section 2, we give some revealing equivalent notions of the idea of autonomous functions—in particular the meromorphic function $y(x)$ is autonomous if and only if x does *not* belong to the differential field generated by $y(x)$. In Section 3, we show that neither the sum nor the product of two autonomous functions need be autonomous. This is not so surprising since being autonomous is an extension of being periodic, and no one expects the sum or product of periodic functions (with different periods!) to be periodic. In Section 4, we use a result of Steinmetz [STE] to prove that if $g(x)$ is an autonomous *entire* function, and if $f(x)$ is *any* differentially algebraic entire function, then $f(g(x))$ is autonomous. (Some restrictions on the domain of f are needed—consider otherwise $g(x)=e^x$ and $f(x)=\log x$.) In some respects, the class AUT of autonomous functions resembles the class PER of periodic functions. This seems to be an analogy worth pursuing.

In Section 5, we show briefly that we may write $x=f(x)+g(x)$ where $f(x)$ and $g(x)$ are periodic meromorphic functions, but that we may not write x as a finite sum of periodic *entire* functions. We conclude, in Section 6, with a few open problems. In the Appendix, we give a short and accessible proof of the Kolchin–Ostrowski theorem that we use in this paper.

2. Equivalent Formulations

We use the following standard notations: $\mathbb{C}$ for the complex numbers ($\mathbb{R}$ for the real numbers), $\mathbb{C}(x)$ for the field of rational functions, $\mathbb{C}\{y\}$ for the ring of differential polynomials with constant coefficients, $\mathbb{C}(x)\{y\}$ for the ring of differential polynomials with coefficients in $\mathbb{C}(x)$, and AUT for the class of autonomous functions.

PROPOSITION 1. *The following are equivalent.*

(i) *u is autonomous.*

(ii) *Let $P(x, y) = \sum P_i(y) x^i$ be in $\mathbb{C}(x)\{y\}$ with $P_i(y) \in \mathbb{C}\{y\}$. If $P(x, u(x)) = 0$ for all x, then $P_i(u(x)) = 0$ for all i and all x.*

(iii) *Let $I = \{P \in \mathbb{C}(x)\{y\}: P(x, u(x)) = 0\}$ be the radical differential ideal of all differential polynomials that annihilate u. Then I has a (finite) basis of autonomous differential polynomials, i.e., elements of $\mathbb{C}\{y\}$.*

(This means that there exist autonomous differential polynomials $P_1, ..., P_m$ such that for any $f \in I$, some power f^N of f lies in the differential ideal generated by $P_1, ..., P_m$. In other words, I is finitely generated as a *radical* differential ideal.)

(iv) *$x \notin \mathbb{C}\langle u \rangle$, the differential field generated by $\mathbb{C}$ and u.*

(v) *u has a minimal differential polynomial (see below) that is autonomous.*

Proof. (i) $\Rightarrow$ (ii). Assume u is autonomous. If $P(x, u(x)) = 0$ then $0 = P(x, u(x+c)) = \sum_{i=0}^{m} P_i(u(x+c)) x^i$ for all x and c. Fix $z \in \mathbb{C}$ and let $x_0, ..., x_m$ be distinct elements of $\mathbb{C}$. Choose $t_0, ..., t_m$ so that $x_j + t_j = z$ for $j = 0, ..., m$. We then have $\sum_{i=0}^{m} P_i(u(z)) x_j^i = 0$ for $j = 0, ..., m$. Using the Vandermonde determinant, we see that this implies that $P_i(u(z)) = 0$ for $i = 0, ..., m$.

(ii) $\Rightarrow$ (*i*). Easy.

(ii) $\Rightarrow$ (iii). Easy. Of course, to make the basis *finite*, one uses the Ritt–Raudenbush basis theorem (see [KAP]).

(iii) $\Rightarrow$ (i). Easy

(ii) $\Rightarrow$ (iv). Assume $x \in \mathbb{C}\langle u \rangle$. We then have $x = P(u)/Q(u)$ for some $P, Q \in \mathbb{C}\{y\}$ with $Q(u) \neq 0$. Therefore $Q(u) x - P(u) = 0$, while $Q(u) \neq 0$, contradicting (ii).

(iv) $\Rightarrow$ (ii). Assume that there are $P_i \in \mathbb{C}\{y\}$ such that $\sum_{i=0}^{m} P_i(u) x^i = 0$ with the $P_i(u)$ not all 0. Among all such relations, select one with the smallest m. Dividing by $P_m(u)$ and differentiating, we find that $(P_{m-1}/P_m)' = -m$. Therefore $x = (-P_{m-1}(u)/mP_m(u)) + c \in \mathbb{C}\langle u \rangle$, contradicting (iv).

We will complete the proof of Proposition 1 after a brief excursion into minimal polynomials.

Among all the non-zero $P \in \mathbb{C}(x)\{y\}$ that annihilate the DA function u, choose one of minimal order (say n), and of minimal degree in $u^{(n)}$. Such a differential polynomial is called a *minimal* differential polynomial for u. We will gather some facts about such polynomials in the following lemma.

Recall that for a differential polynomial of order n, the coefficient of the highest power of $y^{(n)}$ is denoted by I and is called the *initial*, and $\partial P/\partial y^{(n)}$ is denoted by S and is called the *separant*.

LEMMA 1. (a) *If P is a minimal polynomial for f, then P is irreducible as a polynomial in $y^{(n)}$ with coefficients in $\mathbb{C}(x, y, ..., y^{(n-1)})$.*

(b) *If P_1 and P_2 are both minimal polynomials for f, then $P_1 = a(x, y, ..., y^{(n-1)})\, P_2$ for some $a \in \mathbb{C}(x, y, ..., y^{(n-1)})$.*

(c) *If $Q \in \mathbb{C}(x)\{y\}$ annihilates f, then for some non-negative integers i and j, $S^i I^j Q \in [P]$, the differential ideal generated by P in $\mathbb{C}(x)\{y\}$, i.e., the ideal generated by $P, P', P'', ...,$ in $\mathbb{C}(x)\{y\}$.*

(d) *Let $Q \in \mathbb{C}(x)\{y\}$, let P be a minimal polynomial for f, and assume that the order of P is the same as the order of Q and that Q is irreducible over $\mathbb{C}(x, y, ..., y^{(n-1)})$. If Q annihilates f, then Q is a minimal polynomial for f.*

Proof. (a) If $P = QR$, then either $Q[f] = 0$ or $R[f] = 0$. Assume that $Q[f] = 0$. Comparing orders and degrees, we see that $R \in \mathbb{C}(x, y, ..., y^{(n-1)})$, so that P is irreducible as claimed.

(b) We may write

$$P_1 = A_m(y^{(n)})^m + \cdots + A_0$$
$$P_2 = B_m(y^{(n)})^m + \cdots + B_0,$$

where $A_i, B_i \in \mathbb{C}(x)[y, ..., y^{(n-1)}]$. Now $B_m P_1 - A_m P_2$ has smaller degree in $y^{(n)}$, so that $B_m P_1 - A_m P_2 \equiv 0$. Using the irreducibility of P_1 and P_2, we see that P_1 divides P_2 and so $P_1 = aP_2$.

(c) This is obtained by differentiating and using the usual division algorithm in several variables—see [KAP] for details.

(d) Using the division algorithm, we see that there exist $A_0, ..., A_t$ in $\mathbb{C}(x)\{y)$ such that $R = I^i Q - (A_0 P + \cdots + A_t P^t)$ has degree in $y^{(n)}$ less than that of P. Since $R(f) = 0$, we have $R \equiv 0$. Therefore P divides $I^i Q$, so that P divides Q. Since Q is irreducible, we see that $P = aQ$ where $a \in \mathbb{C}(x, y, ..., y^{(n-1)})$, so that Q is therefore a minimal polynomial for f.

We now complete the proof of Proposition 1 by showing that (i) $\Leftrightarrow$ (v). Assume that u is autonomous and let $P(x, y, ..., y^{(n)})$ be a minimal polynomial. We may write $P = \sum x^i P_i$, where each $P_i \in \mathbb{C}[y, ..., y^{(n)}]$. By the proved implication (i) $\Rightarrow$ (ii), we have $P_i(u) = 0$ for each P_i for some i, we know that the order of P_i is n and the degree of P_i in $y^{(n)}$ is the same as the degree of P in $y^{(n)}$. Therefore P_i is a minimal polynomial for f, and is autonomous.

To show that (v) $\Rightarrow$ (i), assume that u has a minimal polynomial P in $\mathbb{C}\{y\}$. Let Q be in $\mathbb{C}(x)\{y\}$ and assume that $Q(u)=0$. By Lemma 1(c), $I^i S^j Q \in [P]$. Since any $R \in [P]$ satisfies $R(f(x+c))=0$ for all $c \in \mathbb{C}$, we have

$$I^i[u(x+c)]\, S^j[u(x+c)]\, Q[u(x+c)]=0$$

for all $c \in \mathbb{C}$. Note that I and S each either has lower order or lower degree (in $y^{(n)}$) than P. So by the minimality of P, we cannot have $I[u(x+c)]$ or $S[u(x+c)]$ vanishing. Therefore $Q[u(x+c)]=0$ for all $c \in \mathbb{C}$, and u is therefore autonomous.

PROPOSITION 2. *Suppose that u satisfies no $(n-1)$st order* ADE *but that it does satisfy an autonomous* ADE *of order n, say $P(y, ..., y^{(n)})=0$ for some $P \in \mathbb{C}\{y\}$. Then u is autonomous.*

Proof. Let Q be an irreducible factor of P in $\mathbb{C}[y, ..., y^{(n)}]$ that involves $y^{(n)}$. Then Q has the same order as a minimal polynomial for u, so by Lemma 1 (d), it must be a minimal polynomial for u. By the part (v) $\Rightarrow$ (i) of Proposition 1, u is autonomous.

Application 1. $u=(1+e^{2x})/(1-e^{2x}) \in \mathrm{AUT}$ because it satisfies $y' = y^2-1$, and of course does not satisfy any ADE of order 0.

Application 2. Consider the ADE

$$y' = y^2(y-1),$$

and note that

$$\frac{1}{y^2(y-1)} = \frac{1}{y-1} - \frac{1}{y} - \frac{1}{y^2},$$

so that

$$\int \frac{y'\,dx}{y^2(y-1)} = \log\left(\frac{y-1}{y}\right) + \frac{1}{y} = x + c.$$

If we let $c=0$, say, and let u be an analytic solution of

$$\log\left(1-\frac{1}{u}\right) + \frac{1}{u} = x,$$

then u is not an elementary function, and is consequently not algebraic. Therefore u is autonomous. That u is not elementary follows from [SIN, Corollary 2].

PROPOSITION 3. *Suppose that u is a C^∞ function that satisfies an nth order* ADE *of the form*

$$p(x)\,y^{(n)}(x) = r(x, y(x), ..., y^{(n-1)}(x)), \qquad (*)$$

where r is a polynomial in its arguments and p is a polynomial in x. Suppose further that u satisfies no $(n-1)$st order ADE. *Let $u^\perp$ be the ideal of all differential polynomials that annihilate u. Write $(*)$ as*

$$Q(x, y(x), ..., y^{(n)}(x)) = 0. \qquad (\#)$$

Then $u^\perp$ is the radical differential ideal generated by Q, i.e., $u^\perp = \{Q\}$.

Here $\{Q\}$ is the ideal of all differential polynomials P, some integer power of which belongs to the differential ideal generated by Q.

Proof of Proposition 3. Solve $(*)$ for $y^{(n)}(x)$ and differentiate successively to get $y^{(n)}(x)$, $y^{(n+1)}(x), ...$ as polynomials in $y(x)$, $y'(x), ..., y^{(n-1)}(x)$ whose coefficients are rational functions of x. Suppose now that $P \in u^\perp$, say $P(x, u(x), u'(x), ..., u^{(N)}(x)) = 0$. Insert the expressions just obtained for $u^{(n)}$, $u^{(n+1)}, ..., u^{(N)}$ and write the result as $\tilde{P}(x, u(x), u'(x), ..., u^{(n-1)}(x)) = 0$. Since u satisfies no $(n-1)$st order (non-trivial) ADE, we must have $\tilde{P} \equiv 0$. Now suppose that v is any C^∞ solution of $Q = 0$. Repeating the above manipulations, with v in place of u, we get

$$P(x, v(x), v'(x), ..., v^{(N)}(x)) = \tilde{P}(z, v(x), v'(x), ..., v^{(n-1)}(x)).$$

Since $\tilde{P} \equiv 0$, we have $P(x, v(x), ..., v^{(N)}(x)) = 0$. By the differentially algebraic form of the Nullstellensatz (see [SEI]), $P \in \{Q\}$ and the proposition is proved.

Note that this gives a second proof of the fact that if u satisfies no $(n-1)$st order ADE, but *does* satisfy an ADE of the form

$$y^{(n)}(x) = r(y(x), y'(x), ..., y^{(n)}(x)), \qquad (\clubsuit)$$

where r is a polynomial that does not involve x, then $u \in$ AUT, since in this case, Q is an autonomous differential polynomial.

We conclude this section with an amusing byplay.

DEFINITION. *A function u is said to be anti-autonomous if for any $c \in \mathbb{C}$, $c \neq 0$, there is an* ADE *$P_c(x, y, ..., y^{(n)}) = 0$ such that $P_c(x, u(x), ..., u^{(n)}(x)) = 0$ and $P_c(x, u(x+c), ..., u^{(n)}(x+c)) \neq 0$.*

Thus, an anti-autonomous function u is one such that *each* of its translates can be distinguished from u by means of differential algebra.

PROPOSITION 4. *Any function that is not autonomous is anti-autonomous.*

Proof. If u is not autonomous, then $x \in \mathbb{C}\langle u \rangle$, by (iv) $\Rightarrow$ (i) of Proposition 1. This means that

$$x = \frac{G(u, u', ..., u^{(n)})}{H(u, u', ..., u^{(n)})}$$

for some G, H in $\mathbb{C}\{y\}$, where $H(u) \not\equiv 0$. Let $P = xH - G$, so that $P(x, u, ..., u^{(n)}) = 0$. If $P(x, u(x+c), ..., u^{(n)}(x+c)) = 0$, then $P(x-c, u(x), ..., u^{(n)}(x)) = 0$, so that $(x-c)\, H(u, ..., u^{(n)}) - G(u, ..., u^{(n)}) = 0$. Thus $H(u, ..., u^{(n)}) \equiv 0$, a contradiction. Notice that one P works for all c. This gives the slickest proof that a periodic function must be autonomous. For since $f(x) = f(x+c)$ for some $c \neq 0$, f cannot be anti-autonomous, and is hence autonomous.

Remark. Proposition 4 indicates that the term "differentially periodic" might be a good substitute for "autonomous."

3. SUMS AND PRODUCTS OF AUTONOMOUS FUNCTIONS

In the following, we shall use the Kolchin–Ostrowski theorem (see [KOL]). In a differential field, a constant is any term whose derivative is zero.

THEOREM K–O. *Let $k \subset K$ be differential fields of characteristic zero with the same constant subfields. Let $u_1, ..., u_n, v_1, ..., v_m$ be elements of K satisfying $u_i' \in k$ for $i = 1, ..., n$ and $v_i'/v_i \in k$ for $i = 1, ..., m$. If $u_1, ..., u_n, v_1, ..., v_m$ are algebraically dependent over k, then either there exist constants $c_1, ..., c_n$, not all zero, such that $c_1 u_1 + \cdots + c_n u_n \in k$ or there exist integers $n_1, ..., n_m$, not all zero, such that $v_1^{n_1} v_2^{n_2} \cdots v_m^{n_m} \in k$.*

We shall use this theorem for the cases corresponding to the following pairs of integers (n, m): $(1, 0)$, $(0, 1)$, $(2, 0)$, and $(1, 1)$, and the reader should restate this theorem in these special cases. Note that we do not assume that the u_i or v_i are distinct. If, for example, $u_1 = u_2$, we may take $c_1 = 1$ and $c_2 = -1$.

We give a simple proof of Theorem K–O in the Appendix. Our proof is in the spirit of [ROS 1]. Another similar, but less elementary, proof is given in [ROS 2].

LEMMA 2. *If u is autonomous and u is not algebraic over $\mathbb{C}\langle u' \rangle$, then $u + x$ is autonomous.*

Proof. If $u+x$ is not autonomous, then $x \in \mathbb{C}\langle u+x\rangle = \mathbb{C}(u+x, u', u'', ...)$. Therefore u and x are algebraically dependent over $\mathbb{C}\langle u'\rangle$. Both x' and u' are in $\mathbb{C}\langle u'\rangle$, so Theorem K–O implies that there are constants c and d, not both zero, such that $cu+dx \in \mathbb{C}\langle u'\rangle \subseteq \mathbb{C}\langle u\rangle$. We must have $d \neq 0$, implying that $x \in \mathbb{C}\langle u\rangle$, a contradiction.

PROPOSITION 5. *There exist two autonomous entire functions f and g such that $f+g$ is not autonomous.*

Proof. Let $f = \int e^{e^x}\,dx$, $g = -\int e^{e^x}\,dx + x$. We then have $f+g=x$, so $f+g$ is not autonomous. We now show that f is autonomous. If $x \in \mathbb{C}\langle \int e^{e^x}\,dx\rangle = \mathbb{C}(\int e^{e^x} dx, e^{e^x}, e^x)$, then x and $\int e^{e^x}\,dx$ would be algebraically dependent over $\mathbb{C}\langle e^{e^x}\rangle$. This would imply that $\int e^{e^x}\,dx$ would be algebraic over $\mathbb{C}\langle x, e^{e^x}\rangle$, contradicting the fact that $\int e^{e^x}\,dx$ is not elementary [ROS1, p. 971]. This fact also implies that $u = \int e^{e^x}\,dx$ is not algebraic over $\mathbb{C}\langle u'\rangle = \mathbb{C}\langle e^{e^x}\rangle$. Of course the same holds for $-\int e^{e^x}\,dx$ so Lemma 2 implies that $-\int e^{e^x}\,dx + x$ is also autonomous.

We can refine the above proof to find entire functions F and G with F autonomous and G periodic such that $F+G=x$. To do this, note that $f = \int e^{e^x}\,dx$ is not periodic but that for some non-zero constant c, $f-cx$ is periodic of period $2\pi i(f(x+2\pi i) - f(x) = d \neq 0$ for some constant d, so let $c = d/2\pi i)$. Let $F = (1/c)f$ and $G = -(1/c)(f-cx)$. In Section 5, we shall show that x is not the sum of a finite number of periodic entire functions.

LEMMA 3. *If f is autonomous, then* $\exp(\int f)$ *is autonomous.*

Proof. If $x \in \mathbb{C}\langle \exp(\int f)\rangle$ then x and $\exp(\int f)$ are algebraically dependent over $\mathbb{C}\langle f\rangle$. Theorem K–O implies that either $x \in \mathbb{C}\langle f\rangle$ or $\exp(N\int f) \in \mathbb{C}\langle f\rangle$ for some positive integer N. The first alternative is a contradiction. The second alternative implies that $\exp(\int f)$ is algebraic over $\mathbb{C}\langle f\rangle$, so x would be algebraic over $\mathbb{C}\langle f\rangle$. Theorem K–O again implies that x would be in $\mathbb{C}\langle f\rangle$, a contradiction.

PROPOSITION 6. *There exist autonomous functions F and G such that FG is not autonomous.*

Proof. Let f and g be as in Proposition 2, and let $F = \exp(\int f)$ and $G = \exp(\int g)$. Then $FG = \exp(\int f+g)$. Since $x \in \mathbb{C}\langle f+g\rangle \subseteq \langle FG\rangle$, FG is not autonomous.

4. Compositions with Autonomous Entire Functions

Theorem 1. *If g is an autonomous entire function and f is any differentially algebraic meromorphic function in $\mathbb{C}$, then $f\circ g$ is autonomous.*

Proof. Our proof utilizes the following theorem of Steinmetz (see [STE, GRO]). We use the standard concepts and notation of Nevanlinna's theory of value distribution (see [NEV]).

Theorem S. *Let $F_0, F_1, ..., F_m (m\geqslant 1)$ be meromorphic functions on $\mathbb{C}$, none of which vanishes identically. Let $h_0, h_1, ..., h_m$ be arbitrary meromorphic functions on $\mathbb{C}$. Let g be a nonconstant entire function and suppose that*

$$\sum_{\mu=0}^{m} T(r, h_\mu)\leqslant KT(r, g)+S(r, g), \tag{0}$$

where K is a positive constant. Further, suppose that

$$F_0(g)\, h_0+F_1(g)\, h_1+\cdots+F_m(g)\, h_m\equiv 0. \tag{1}$$

Then there exist polynomials $P_0, P_1, ..., P_m$, none of which vanishes identically, so that

$$P_0(g)h_0+P_1(g)\, h_1+\cdots+P_m(g)h_m\equiv 0. \tag{1'}$$

To start our proof of Theorem 1, we write $h=f\circ g$ (i.e., $h(x)=f(g(x))$, and suppose that h is *not* autonomous. By Proposition 1, $x\in\mathbb{C}\langle h\rangle$, i.e., $x=-P(\mathbf{h})/Q(\mathbf{h})$, where P and Q are autonomous differential polynomials in h, and $Q(h)\not\equiv 0$. Thus

$$xQ(\mathbf{h}(x))=-P(\mathbf{h}(x)). \tag{2}$$

Now, as remarked in [STE], we may write, for any non-negative integer ν

$$h^{(\nu)}(x)=\sum_{j=1}^{\nu} f^{(j)}(g(x))\, D_{\nu j}[g(x)], \tag{3}$$

where each $D_{\nu j}$ is a homogeneous differential polynomial of degree j with constant coefficients. We write

$$Q[h]=\sum (N_\mu[f]\circ g)H_\mu[g]$$

$$P[h]=\sum{}' (N_\mu[f]\circ g)H_\mu[g],$$

where $\sum$ and $\sum'$ are sums over different finite index sets. For a dependent differential variable $u(=u(x))$ we write, for g fixed as above,

$$\tilde{Q}[u]=\sum (N_\mu[f]\circ g)\, H_\mu[u],$$

$$\tilde{P}[u]=\sum{}' (N_\mu[f]\circ g)\, H_\mu[u],$$

$$\tilde{\Delta}[u]=\tilde{P}[k]+x\tilde{Q}[u].$$

We consider $\tilde{Q}$ as a differential polynomial in u with coefficients in the differential ring $\mathscr{R}$ generated by the constants and the functions $N_\mu[f]\circ g$. This ring is a Ritt ring, i.e., a differential ring that contains the rational numbers $\mathbb{Q}$. Let M be an *autonomous* minimal differential polynomial for g, whose existence is guaranteed by Proposition 1 (v). Let I be the initial of M and S the separant of M. By the division algorithm (see [KAP, p. 46, Lemma 7.3]) there are non-negative integers α and β such that

$$T \doteq I^\alpha S^\beta \tilde{Q} \equiv L \bmod \{M\},$$

in $\mathscr{R}\{u\}$, where L is lower than M, i.e., L either has lower order than M or lower degree. Here $\{M\}$ denotes the differential ideal generated by M in $\mathscr{R}\{u\}$. Remember that we are here considering M, T, Q, L, etc., as elements of $\mathscr{R}\{u\}$. Note that $L[g]\neq 0$, for this would make $\tilde{Q}[g]=0$ (and hence $Q[g]=0$) since $I[g]\neq 0$ and $S[g]\neq 0$ because both I and S are lower than M. Define

$$W[u]=I^\alpha S^\beta P[u]+xL[u].$$

Then $W[g]=0$. We may write

$$W[g]=\sum F_\mu(g)\tilde{H}_\mu[g]+x\sum{}' F_\mu(g)\,\tilde{H}_\mu[g]=0, \qquad (*)$$

where the F_μ are certain entire functions, and the $\tilde{H}_\mu$ are entire expressions in g, g', g'', ... Note that, because M is autonomous, $I^\alpha S^\beta P$ and L do not involve the independent variable x. We may now apply Theorem S to get *polynomials* P_μ such that

$$\sum P_\mu(g)\,\tilde{H}_\mu[g]+x\sum{}' P_\mu(g)\,\tilde{H}_\mu[g]=0.$$

It is clear that, except in the trivial case where g is a constant, it cannot be a polynomial, for if g were a non-constant polynomial in x then we could take several derivatives of g to get $x\in\mathbb{C}\langle g\rangle$, contradicting g being autonomous. Thus, $T(r,x)=O(T(r,g))$. Since $T(r,xl(x))\leqslant T(r,l(x))+$

$T(r, x)$, we see that the hypothesis (0) of Theorem S is fulfilled in our case, using the known estimate

$$T(r, H_\mu[g]) = O(T(r, g)) + S(r, g)$$

(see [CLU]). Notice now that $\sum' P_\mu(\cdot)\, \tilde{H}_\mu[\cdot]$ is lower than M since each of the $\tilde{H}_\mu$, as differential expressions occurring in L, is lower than M and since the P_μ are only ordinary polynomials, and M is not of order 0. Hence $\sum' P_\mu(g)\, \tilde{H}_\mu[g] \neq 0$. Thus $(*)$ may be rewritten as

$$P^*[g] + xQ^*[g] = 0, \qquad \text{where } Q^*[g] \neq 0, \tag{\#}$$

which contradicts the hypothesis that g is autonomous. This contradiction proves the theorem.

Note that the results we have proved can be used to generate a large number of autonomous entire functions. Start with the class of all periodic entire functions. Then, for every f we have so far, adjoin $\exp(\int f)$. Now every derivative of an autonomous function g must be autonomous, since if $x \in \mathbb{C}\langle g' \rangle$, then $x \in \mathbb{C}\langle g \rangle$. Thus, we may adjoin all derivatives of the autonomous functions constructed so far. Also, for every f we have so far, we may adjoin $F(f)$, where F is any differentially algebraic entire function. Keep on repeating these processes to get a substantial class of entire autonomous functions.

5. Finite Sums of Periodic Functions

In this section, we will write z, instead of x, for the independent variable.

PROPOSITION 7. *There exist two differentially algebraic and periodic meromorphic functions in $\mathbb{C}$ such that $z = f(z) + g(z)$ for all z. On the other hand, z is not the sum of finitely many periodic entire functions.*

Proof. We use the Weierstrass ζ-function $\zeta(z)$ using [SAZ, Chap. VIII, Sect. 6] as a reference. We are using the "periods" $\{\omega_k\} = \{l + mi : l, m \in \mathbb{Z}, l^2 + m^2 \neq 0\}$ so that

$$\zeta(z) = \frac{1}{z} + \sum_{\substack{l, m \in \mathbb{Z} \\ l^2 + m^2 \neq 0}} \left[\frac{1}{z - (l + mi)} + \frac{1}{l + mi} + \frac{z}{(l + mi)^2} \right]. \tag{♥}$$

Further, we have

$$\zeta(z + 1) - \zeta(z) = \eta, \qquad \zeta(z + i) - \zeta(z) = \eta',$$

where η and η' are constants that satisfy Legendre's equation

$$i\eta - \eta' = 2\pi i.$$

It is clear from (♥) that η is real, and hence $\eta \neq \eta'$. Let

$$\tilde{f}(z) = \zeta(z) - \eta z, \qquad -\tilde{g}(z) = \zeta(z) - \eta' z.$$

Then $\tilde{f}$ is periodic of period 1, and $\tilde{g}$ is periodic of period i. Furthermore, $\tilde{f}(z) + \tilde{g}(z) = (\eta' - \eta)\, z$, and the first part of the result follows on choosing $f(z) = (\eta' - \eta)^{-1} \tilde{f}(z)$ and $g(z) = (\eta' - \eta)^{-1}\, \tilde{g}(z)$.

Note that this result gives a meromorphic version of Proposition 5, since g and h, being periodic, must be autonomous.

For the second part, let us suppose that

$$z = f_1(z) + \cdots + f_n(z), \tag{♣}$$

where each $f_i(z)$ is entire, with period ω_i. By coalescing terms if necessary, we may suppose that the ω_i are pairwise incommensurate.

LEMMA 4. *If f is an entire function and if ω and ω' are two incommensurable complex numbers such that*

$$f(z+\omega) - f(z) = 0 \qquad \textit{and} \qquad f(z+\omega') - f(z) = c, \tag{♦}$$

then $c = 0$ and f is a constant.

Proof. Differentiate (♦) to see that $f'(z)$ is a doubly periodic entire function, and hence a constant. Thus $f(z) \equiv Az + B$, and since f has period ω, A must equal 0 and $f(z) \equiv B$, a constant.

Suppose now that (♣) holds. For a complex number ω, let Δ_ω be the difference operator with spacing ω: $(\Delta_\omega * f)(z) = f(z+\omega) - f(z)$. Choose a representation (♣) with n as small as possible. Clearly, $n \geqslant 2$, since z is not periodic. Let us now apply the operator $\Delta = \Delta_{\omega_1} * \Delta_{\omega_2} * \cdots * \Delta_{\omega_{n-1}}$ to (♣) to get

$$(\Delta_{\omega_n} * f_n)(z) = 0,$$
$$(\Delta * f_n)(z) = 0.$$

If $n = 2$, then $\Delta = \Delta_{\omega_1}$, and we may apply Lemma 4 to see that f_n is a constant. We could add this constant to f_1 and reduce the number n in the representation (♣). Therefore $n \geqslant 3$. Let

$$F(z) = (\Delta_{\omega_1} * \Delta_{\omega_2} * \cdots * \Delta_{\omega_{n-2}} * f_n)(z).$$

Then

$$(\Delta_{\omega_n} * F)(z) = 0, \qquad (\Delta_{\omega_{n-1}} * F)(z) = 0.$$

Applying Lemma 4 again, we see that $F(z)$ is a constant. Repeating this argument, we eventually see that $f_n(z)$ is a constant, which leads to a contradiction as above.

6. Open Problems

Problem 1. If $f(z)$ and $g(z)$ are autonomous analytic functions on suitable domains, must $f(g(z))$ also be autonomous? (Note: following Proposition 1 we could now take as our *definition* of f being autonomous, where f is only defined on an open subset of $\mathbb{C}$, that $z \notin \mathbb{C}\langle f \rangle$.)

Problem 2. Can an autonomous function have the unit circle as a natural boundary?

Problem 3. Does z belong to the *ring* generated by the periodic entire functions? That is, can we write z as a finite sum of finite products of periodic entire functions? (Note that z does belong to the *field* generated by the periodic entire functions. This follows from Proposition 7 and the fact that every periodic function that is meromorphic in $\mathbb{C}$ is the quotient of two periodic entire functions. For (see [SAZ, Chap. 8]), if $f(z)$ is a periodic meromorphic function on the plane, say with period $2\pi i$, then (and only then) we may write $f(z) = M(e^z)$, where $M(w)$ is meromorphic in the set $\ddot{S}$, which is the Riemann sphere with 0 and ∞ deleted. But, by the Mittag-Leffler theorem, every meromorphic function M on $\ddot{S}$ may be written as the quotient $M = A/B$ of two holomorphic functions on $\ddot{S}$. Then $f(z) = A(e^z)/B(e^z)$ is the desired representation.)

Problem 4. Does there exist an entire (or meromorphic) autonomous function that is of order 0? Of fractional order? Of integer order $\neq 1$ ($\neq 1$ or 2 in the meromorphic case)?

APPENDIX: A Simple Proof of Theorem K–O

We restate this theorem for the reader's convenience.

Theorem K–O. *Let $k \subset K$ be differential fields of characteristic zero with the same constant subfields. Let $u_1, \ldots, u_n, v_1, \ldots, v_m$ be elements of K satisfying $u' \in k$ for $i = 1, \ldots, n$ and $v'/v \in k$ for $i = 1, \ldots, m$. If*

$u_1, \ldots, u_n, v_1, \ldots, v_m$ are algebraically dependent over k, then either there exist constants $c_1, \ldots, c_n$, not all zero, such that $c_1 u_1 + \cdots c_n u_n \in k$ or there exist integers $n_1, \ldots, n_m$, not all zero, such that $v_1^{n_1} v_2^{n_2} \cdots v_m^{n_m} \in k$.

This result will follow from the next two lemmas.

LEMMA A1. *Let $k \subset K$ be differential fields of characteristic 0 with the same constant subfields. Let $y \in K$ and assume y is algebraic over k*

(a) *if $y' \in k$ then $y \in k$;*

(b) *if $y'/y \in k$ then $y^N \in k$ for some non-zero integer N.*

Proof. (a) Let $y^m + a_{m-1} y^{m-1} + \cdots + a_0$ be the minimal polynomial for y over k. Assuming $m > 1$, we will derive a contradiction. Differentiating, we have

$$(my' + a'_{m-1}) y^{m-1} + \cdots + (a_1 y' + a'_0) = 0.$$

If $m > 1$, we must have $my' + a'_{m-1} = 0$. This implies that $y + (1/m) a_{m-1}$ is a constant and so lies in k. Therefore $y \in k$.

(b) Let $y'/y = t \in k$. Let $y^m + a_j y^j + \cdots + a_0 = 0$ be the minimal polynomial for y with $a_j \neq 0$. Differentiating we have

$$my^m t + (a'_j + jt) y^j + \cdots + a'_0 = 0.$$

Comparing this equation to our original equation, we see that $mta_j = a'_j + jt$. Therefore $(m-j) t = a'_j/a_j$ and so

$$(m-j) \frac{y'}{y} - \frac{a'_j}{a_j} = 0.$$

This implies that

$$\frac{(y^{m-j}/a_j)'}{y^{m-j}/a_j} = 0$$

so y^{m-j}/a_j is a constant. Therefore $y^{m-j} \in k$.

LEMMA A2. *Let $k \subset K$ be differential fields of characteristic 0 with the same constant subfields. Let $w \in K$ and assume w is transcendental over k. Let $z \in K$ be algebraic over $k(w)$.*

(a) *If $w' \in k$ and $z' \in k$, then there is a constant c such that $cw + z \in k$.*

(b) *If $w' \in k$ and $z'/z \in k$, then $z^n \in k$ for some non-zero integer n.*

(c) *If $w'/w \in k$ and $z'/z \in k$, then $z^n w^m \in k$ for some integers n, m, not both zero.*

Proof. (a) Since z is algebraic over $k(w)$ and $z' \in k(w)$, Lemma A1 implies that $z \in k(w)$. We expand z in partial fractions

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{q_{ij}(w)}{(p_i(w))^j} + h(w),$$

where the p_i are monic irreducible polynomials in w, $\deg q_{ij} < \deg p_i$ and the q_{ij} and h are polynomials. Fix some $p_i(w)$ and call it $p(w)$ and let $n_i = n$. Differentiating, we get

$$\begin{aligned} z' &= \left(\frac{q_n(w)}{(p(w))^n} + \cdots\right)' \\ &= \frac{-nq_n(w)(p(w))'}{p^{n+1}(w)} + \begin{array}{l}\text{terms whose denominators}\\ \text{contain lower powers of } p.\end{array} \end{aligned}$$

Since $p(w)$ is monic, $\deg(p(w))' < \deg(p(w))$. Therefore $p(w)$ does not divide $-n\, q_n(w)(p(w))'$. This implies that $p(w)$ actually appears in the denominator of the partial fraction decomposition of z'. Since $z' \in k$, this is a contradiction unless

$$z = h(w)$$

for some polynomial $h(w) = a_m w^m + \cdots + a_0$. Differentiating, we find

$$z' = a'_m w^m + (ma_m w' + a'_{m-1})\, w^{m-1} + \cdots a'_0.$$

If $m > 1$, then $a'_m = 0 = ma_m w' + a'_{m-1}$. This implies that $w' = (-a_{m-1}/ma_m)'$ so $(w + a_{m-1}/ma_m)' = 0$. We could conclude that $w \in k$, a contradiction. Therefore $m \leqslant 1$, so $z = a_1 w + a_0$. Differentiating again, we find

$$z' = a'_1 w + (a_1 w' + a'_0).$$

We see that $a'_1 = 0$ (i.e., a_1 is constant) and $z - a_1 w \in k$.

(b) Since z is algebraic over $k(w)$ and $z'/z \in k$, we have by Lemma A1, $z^N \in k(w)$ for some non-zero integer N. We may write

$$z^N = a\, \Pi\, p_i^{n_i},$$

where the p_i are irreducible monic polynomials, $a \in k$ and $n_i \in \mathbb{Z}$. We then have

$$N \frac{z'}{z} = \frac{a'}{a} + \sum n_i \frac{p'_i}{p_i}.$$

Since $z'/z \in k$ and since $\deg p_i' < \deg p_i$, uniqueness of the partial fraction decomposition lets us conclude that each $n_i = 0$, so $z^N = a \in k$.

(c) We first note that if $p(w)$ is a monic irreducible polynomial and $p(w) \neq w$, then $p(w)$ does not divide $(p(w))'$. To see this let $p(w) = w^n + bw^m + \cdots$, b a non-zero element of k and $n > m$. Differentiating, we have

$$(p(w))' = ntw^n + (b' + mtb)\, w^m + \cdots,$$

where $t = w'/w$. If $p(w)$ divides $(p(w))'$ then

$$ntb = b' + mtb$$

so

$$(n-m)\, t = \frac{b'}{b}.$$

This implies that $(w^{n-m}/b)'/(w^{n-m}/b) = 0$ or w^{n-m}/b is a constant, contradicting the fact that w is transcendental over k.

Since z is algebraic over $k(w)$ and $z'/z \in k$, Lemma A1 implies that $z^N \in k$ for some non-zero integer N. We may write

$$z^N = aw^M \Pi p_i^{n_i},$$

where the p_i are irreducible monic polynomials $\neq w$, $a \in k$, and M and the n_i are integers. We then have

$$N\frac{z'}{z} = \frac{a'}{a} + Mt + \sum n_i \frac{p_i'}{p_i}.$$

Since p_i does not divide p_i', the uniqueness of the partial fraction decomposition implies that each $n_i = 0$. Therefore $z^N = aw^M$ or $z^N w^{-M} \in k$.

Proof of Theorem K–O. We proceed by induction on $n+m$, If $n+m=1$, then the result is just Lemma A1. Assume $n+m>1$. If $n \neq 0$, we have that $u_2, ..., u_n, v_1, ..., v_m$ are algebraically dependent over $k(u_1)$. By induction, we have that either $c_2u_2 + \cdots + c_nu_n \in k(u_1)$ for some constants $c_2, ..., c_n$ not all zero or $v_1^{n_1} \cdot \cdots \cdot v_m^{n_m} \in k(u_1)$ for some integers $n_1, ..., n_m$ not all zero. In the first case let $z = c_2u_2 + \cdots + c_nu_n$ and $w = u_1$.

If w is algebraic over k, then Lemma A1 implies that $1 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n \in k$. If w is not algebraic over k, apply Lemma A2 to conclude that $w + cz = u_1 + cc_1u_2 + \cdots + cc_n u_n \in k$. In the second case let $z = v_1^{n_1} \cdot \cdots \cdot v_m^{n_m}$ and $w = u_1$. If w is algebraic over k, we argue as above. If w is not algebraic over k, Lemma A2 implies that $z^N = v_1^{Nn_1} \cdot \cdots \cdot v_m^{Nn_m} \in k$. If $n=0$ but $m \neq 0$, we argue in a similar manner using Lemma A2 (c).

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