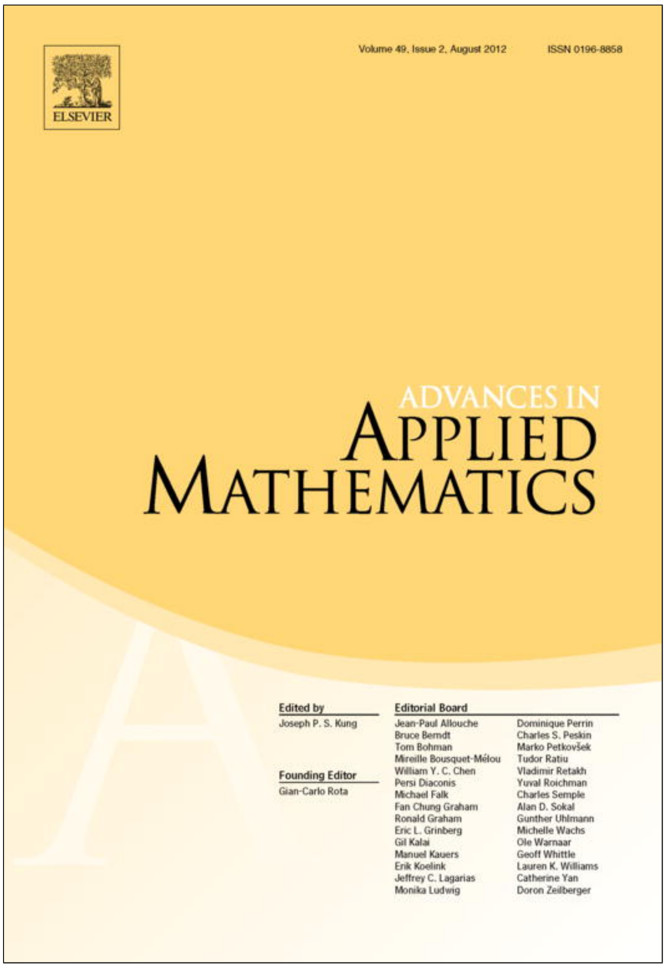
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Residues and telescopers for bivariate rational functions ☆

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Dedicated to the memories of Philippe Flajolet and Herbert S. Wilf

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ABSTRACT

We give necessary and sufficient conditions for the existence of telescopers for rational functions of two variables in the continuous, discrete and q-discrete settings and characterize which operators can occur as telescopers. Using this latter characterization, we reprove results of Furstenberg and Zeilberger concerning diagonals of power series representing rational functions. The key concept behind these considerations is a generalization of the notion of residue in the continuous case to an analogous concept in the discrete and q-discrete cases.

1. Introduction

Residues have played a ubiquitous and important role in mathematics and their use in combinatorics has had a lasting impact (e.g., [26]). In this paper we will show how the notion of residue and its generalizations lead to new results and a recasting of known results concerning telescopers in the continuous, discrete and q-discrete cases.

As an introduction to our point of view and our results, let us consider the problem of finding a differential telescoper for a rational function of two variables. Let k be a field of characteristic zero, $k(t,x)$ the field of rational functions of two variables and $D_t = \partial/\partial_t$ and $D_x = \partial/\partial_x$ the usual derivations with respect to t and x, respectively. Given $f \in k(t,x)$, we wish to find a nonzero operator $L \in k(t)\langle D_t\rangle$, the ring of linear differential operators in D_t with coefficients in $k(t)$, and an element $g \in k(t,x)$ such that $L(f) = D_x(g)$. We may consider f as an element of $\overline{K}(x)$ where $\overline{K}$ is the algebraic closure of $K = k(t)$. As such, we may write

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$$f = p + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j}, \tag{1}$$

where $p \in K[x]$, the β_i are the roots in $\overline{K}$ of the denominator of f and the $\alpha_{i,j}$ are in $\overline{K}$. Note that the element $\alpha_{i,1}$ is the usual *residue* of f at β_i. Using Hermite reduction ([14, p. 39] or Section 2.1 below), one sees that a rational function $h \in K(x)$ is of the form $h = D_x(g)$ for some $g \in K(x)$ if and only if all residues of h are zero. Therefore to find a telescoper for f it is enough to find a nonzero operator $L \in K\langle D_t\rangle$ such that $L(f)$ has only zero residues. For example assume that f has only simple poles, i.e., $f = \frac{a}{b}, a, b \in K[x]$, $\deg_x a < \deg_x b$ and b squarefree. We then know that the Rothstein–Trager resultant [49,45]

$$R := \text{resultant}_x\big(a - zD_x(b), b\big) \in K[z]$$

is a polynomial whose roots are the residues at the poles of f. Given a squarefree polynomial in $K[z] = k(t)[z]$, differentiation with respect to t and elimination allow one to construct a nonzero linear differential operator $L \in k(t)\langle D_t\rangle$ such that L annihilates the roots of this polynomial. Applying L to each term of (1) one sees that $L(f)$ has zero residues at each of its poles. Applying Hermite reduction to $L(f)$ allows us to find a g such that $L(f) = D_x(g)$.

The main idea in the method described above is that nonzero residues are the obstruction to being the derivative of a rational function and one constructs a linear operator to remove this obstruction. This idea is the basis of results in [16] where it is shown that the problem of finding differential telescopers for rational functions in m variables is equivalent to the problem of finding telescopers for algebraic functions in $m-1$ variables and where a new algorithm for finding telescopers for algebraic functions in two variables is given.

For a precise problem description, let $k(t, x)$ be as above and D_t and D_x be the derivations defined above. We define shift operators S_t and S_x as

$$S_t\big(f(t, x)\big) = f(t + 1, x) \quad \text{and} \quad S_x\big(f(t, x)\big) = f(t, x + 1)$$

and q-shift operators (for $q \in k$ not a root of unity) Q_t and Q_x as

$$Q_t\big(f(t, x)\big) = f(qt, x) \quad \text{and} \quad Q_x\big(f(t, x)\big) = f(t, qx).$$

Let Δ_x and $\Delta_{q,x}$ denote the difference and q-difference operators $S_x - 1$ and $Q_x - 1$, respectively. In this paper, we give a solution to the following problem

Existence Problem for Telescopers. *For any $\partial_t \in \{D_t, S_t, Q_t\}$ and $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$ find necessary and sufficient conditions on elements $f \in k(t, x)$ that guarantee the existence of a nonzero linear operator $L(t, \partial_t)$ in ∂_t with coefficients in $k(t)$ (a* telescoper*) and an element $g \in k(t, x)$ (a* certificate*) such that*

$$L(t, \partial_t)(f) = \partial_x(g).$$

As we have shown above, when $\partial_t = D_t$ and $\partial_x = D_x$, a telescoper and certificate exist for any $f \in k(t, x)$. This is not necessarily true in the other cases. In the case when $\partial_t = S_t$ and $\partial_x = \Delta_x$, Abramov and Le [8] showed that there is no telescoper for the rational function $1/(t^2 + x^2)$ and presented a necessary and sufficient condition for the existence of telescopers. Later, Abramov gave a general criterion for the existence of telescopers for hypergeometric terms [6]. The q-analogs were achieved in the works by Le [37] and by Chen et al. [17]. Our approach in this paper represents a unified way of solving the Existence Problem for Telescopers (for bivariate rational functions) in these and the remaining cases. In particular, we will first identify in each case the appropriate notion of “residues” which will be elements of $\overline{k(t)}$, the algebraic closure of $k(t)$. We will show that for any $f \in k(t,x)$ and $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$, there exists a $g \in k(t,x)$ such that $f = \partial_x(g)$ if and only if all the "residues" vanish. We will then show that to find a telescoper, it is necessary and sufficient to find an operator $L(t, \partial_t)$ that annihilates all of the residues.

This necessary and sufficient condition has several applications. For example, our results reduce the Existence Problem for Telescopers to the problem of finding necessary and sufficient conditions that guarantee the existence of operators that annihilate algebraic functions and we present a solution to this latter problem. Our approach also gives termination criteria for the Zeilberger method [9,55,56] and also a strategy for finding telescopers and certificates, which has been successfully used in the continuous case in [16]. In addition, these criteria together with the results in [33,46] can be used to determine if indefinite sums and integrals satisfy (possibly nonlinear) differential equations (see Example 4.10).

The rest of the paper is organized as follows. In Section 2 we define the notions of residues relevant to the discrete and q-discrete cases and show that for any $f \in k(t,x)$ and $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$, there exists a $g \in k(t,x)$ such that $f = \partial_x g$ if and only if all the residues vanish. In Section 3 we characterize those algebraic functions in $\overline{k(t)}$ for which there exist annihilating linear operators $L(t, S_t)$ or $L(t, Q_t)$ as well as prove some ancillary results useful in succeeding sections. In Section 4, we solve the Existence Problem for Telescopers as well as characterize when a linear operator is a telescoper. Using this latter characterization, we can give a proof, using our approach, of the theorem of Furstenberg [29] stating that the diagonal of a rational power series in two variables is an algebraic function. We also discuss a recent example of Ekhad and Zeilberger [25] in the context of the results of this paper. Appendix A contains proofs of the characterizations stated in Section 3.

2. Residues

Let K be a field of characteristic zero and $K(x)$ be the field of rational functions in x over K. Let $\overline{K}$ denote the algebraic closure of K. Let $q \in K$ be such that $q^i \neq 1$ for any nonzero $i \in \mathbb{Z}$, i.e., q is not a root of unity. As in the Introduction, we define the derivation D_x, shift operator S_x, and q-shift operator Q_x on $K(x)$, respectively, as

$$D_x\big(f(x)\big) = \frac{d(f(x))}{dx}, \qquad S_x\big(f(x)\big) = f(x+1), \quad \text{and} \quad Q_x\big(f(x)\big) = f(qx)$$

for all $f \in K(x)$. Let Δ_x and $\Delta_{q,x}$ denote the difference and q-difference operators $S_x - 1$ and $Q_x - 1$, respectively. A rational function $f \in K(x)$ is said to be *rational integrable* (resp. *summable*, *q-summable*) in $K(x)$ if there exists $g \in K(x)$ such that $f = D_x(g)$ (resp. $f = \Delta_x(g)$, $f = \Delta_{q,x}(g)$). This section is motivated by the well known result (Proposition 2.2 below) that characterizes rational integrability in terms of vanishing residues. In the remainder of this section we describe other types of "residues" and how they can be used to give necessary and sufficient conditions for summability and q-summability.

2.1. Continuous residues

Let $f = a/b \in K(x)$ with $a, b \in K[x]$ and $\gcd(a,b) = 1$. Then f can be uniquely written in its partial fraction decomposition

$$f = p + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j}, \tag{2}$$

where $p \in K[x]$, $m, n_i \in \mathbb{N}$, $\alpha_{i,j}, \beta_i \in \overline{K}$, and β_i's are roots of b. From any of the usual proofs of partial fraction decompositions, one sees that all the $\alpha_{i,j}$'s are in $K(\beta_1, \ldots, \beta_m)$.

Definition 2.1 *(Continuous residue).* Let $f \in K(x)$ be of the form (2). The value $\alpha_{i,1} \in \overline{K}$ is called the *continuous residue* of f at β_i (with respect to x), denoted by $\operatorname{cres}_x(f, \beta_i)$.

Note that the continuous residue is just the usual residue in complex analysis. We will define other kinds of residues below but when we refer to a residue without further modification, we shall mean the continuous residue. Although the following is well known (see [50, Proposition 2.1]) we include it since this result is the motivation and model for the considerations that follow.

Proposition 2.2. *Let $f = a/b \in K(x)$ be such that $a, b \in K[x]$ and $\gcd(a, b) = 1$. Then f is rational integrable in $K(x)$ if and only if the residue $\operatorname{cres}_x(f, \beta)$ is zero for any root $\beta \in \overline{K}$ of b.*

Proof. Suppose that f is rational integrable in $K(x)$, i.e., $f = D_x(g)$ for some g in $K(x)$. Writing g in its partial fraction decomposition and differentiating each term, one sees that all the residues of $D_x(g)$ are 0. Conversely, if all residues of f at its poles are zero, then f can be written as

$$f = p + \sum_{i=1}^{m} \sum_{j=2}^{n_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j},$$

where $p \in K[x]$, $\alpha_{i,j}, \beta_i \in \overline{K}$, and β_j's are roots of b. Note that any polynomial is rational integrable in $K(x)$, and for all i, j with $1 \leqslant i \leqslant m$ and $2 \leqslant j \leqslant n_i$,

$$\frac{\alpha_{i,j}}{(x - \beta_i)^j} = D_x\left(\frac{(1 - j)^{-1}\alpha_{i,j}}{(x - \beta_i)^{j-1}}\right).$$

Then $f = D_x(g)$, where g is of the form

$$g = \tilde{p} + \sum_{i=1}^{m} \sum_{j=2}^{n_i} \frac{(1 - j)^{-1}\alpha_{i,j}}{(x - \beta_i)^{j-1}} \quad \text{for some } \tilde{p} \in K[x].$$

For each irreducible factor p of b, the sum in g is a symmetric function of those β_i's that are roots of p. From this one concludes that g lies in $K(x)$. Thus, f is rational integrable in $K(x)$. □

2.2. Discrete residues

Given a rational function, Matusevich [38] found a necessary and sufficient condition for its rational summability. Moreover, one can algorithmically decide whether a rational function is rational summable or not using methods in [2,3,7,5,4,42–44]. Here, we present a rational summability criterion via a discrete analogue of residues. To this end, we first recall some terminology from [2,42] and [51, Chapter 2].

For an element $\alpha \in \overline{K}$, we call the subset $\alpha + \mathbb{Z}$ the *$\mathbb{Z}$-orbit* of α in $\overline{K}$, denoted by $[\alpha]$. For a polynomial $b \in K[x] \setminus K$, the value

$$\max\left\{i \in \mathbb{Z} \;\middle|\; \exists \alpha, \beta \in \overline{K} \text{ such that } i = \alpha - \beta \text{ and } b(\alpha) = b(\beta) = 0\right\}$$

is called the *dispersion* of b with respect to x, denoted by $\operatorname{disp}_x(b)$. The polynomial b is said to be *shift-free* with respect to x if $\operatorname{disp}_x(b) = 0$. Let $f = a/b \in K(x)$ be such that $a, b \in K[x]$ and $\gcd(a, b) = 1$. Over the field $\overline{K}$, f can be decomposed into the form

$$f = p + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x - (\beta_i + \ell))^j}, \tag{3}$$

where $p \in K[x]$, $m, n_i, d_{i,j} \in \mathbb{N}$, $\alpha_{i,j,\ell}, \beta_i \in \overline{K}$, and β_i's are in distinct $\mathbb{Z}$-orbits.

Definition 2.3 *(Discrete residue).* Let $f \in K(x)$ be of the form (3). The sum $\sum_{\ell=0}^{d_{i,j}} \alpha_{i,j,\ell}$ is called the *discrete residue* of f at the $\mathbb{Z}$-orbit $[\beta_i]$ of multiplicity j (with respect to x), denoted by $\mathrm{dres}_x(f, [\beta_i], j)$.

Lemma 2.4. *Let $f = \sum_{\ell=0}^{d} \alpha_\ell/(x-(\beta+\ell))^s$ be such that $d, s \in \mathbb{N}$ and $\alpha_\ell, \beta \in \overline{K}$. Then f is rational summable in $\overline{K}(x)$ if and only if the sum $\sum_{\ell=0}^{d} \alpha_\ell$ is zero that is, if and only if $\mathrm{dres}_x(f, [\beta], s) = 0$.*

Proof. Suppose that the sum $\sum_{\ell=0}^{d} \alpha_\ell$ is zero. We show that f is rational summable in $\overline{K}(x)$. To this end, we proceed by induction on d. In the base case when $d = 0$, f is clearly rational summable in $\overline{K}(x)$ since $f = 0$. Suppose that the assertion holds for $d = m$ with $m \geqslant 0$. Note that

$$\frac{\alpha_{m+1}}{(x-(\beta+m+1))^s} = \Delta_x\left(-\frac{\alpha_{m+1}}{(x-(\beta+m+1))^s}\right) + \frac{\alpha_{m+1}}{(x-(\beta+m))^s}.$$

This implies that

$$\sum_{\ell=0}^{m+1} \frac{\alpha_\ell}{(x-(\beta+\ell))^s} = \Delta_x\left(-\frac{\alpha_{m+1}}{x-(\beta+m+1)^s}\right) + \sum_{\ell=0}^{m} \frac{\tilde{\alpha}_\ell}{(x-(\beta+\ell))^s},$$

where $\tilde{\alpha}_\ell = \alpha_\ell$ if $0 \leqslant \ell \leqslant m-1$ and $\tilde{\alpha}_m = \alpha_{m+1} + \alpha_m$. By definition, the sum $\sum_{\ell=0}^{m} \tilde{\alpha}_\ell$ is still zero. The induction hypothesis then implies that there exists $\tilde{g} \in \overline{K}(x)$ such that

$$\sum_{\ell=0}^{m} \frac{\tilde{\alpha}_\ell}{(x-(\beta+\ell))^s} = \Delta_x(\tilde{g}).$$

So $f = \Delta_x(g)$ with $g = \tilde{g} - \alpha_{m+1}/(x-(\beta+m+1))^s \in \overline{K}(x)$. For the opposite implication, we assume to the contrary that the sum $\sum_{\ell=0}^{d} \alpha_\ell$ is nonzero. Without loss of generality, we can assume that $\alpha_0 \neq 0$. Write $\alpha_0 = \bar{\alpha}_0 + \tilde{\alpha}_0$ such that $\tilde{\alpha}_0 + \sum_{\ell=1}^{d} \alpha_\ell = 0$. Since $\sum_{\ell=0}^{d} \alpha_\ell \neq 0$, $\bar{\alpha}_0 \neq 0$. By the assertion shown above, there exists $\tilde{g} \in \overline{K}(x)$ such that

$$f = \frac{\bar{\alpha}_0}{(x-\beta)^s} + \Delta_x(\tilde{g}).$$

Since $\mathrm{disp}_x((x-\beta)^s) = 0$ and $\bar{\alpha}_0 \neq 0$, $\bar{\alpha}_0/(x-\beta)^s$ is not rational summable by [38, Lemma 3] or [33, Lemma 6.3]. Then f is not rational summable in $\overline{K}(x)$. This completes the proof. □

Proposition 2.5. *Let $f = a/b \in K(x)$ be such that $a, b \in K[x]$ and $\gcd(a, b) = 1$. Then f is rational summable in $K(x)$ if and only if the discrete residue $\mathrm{dres}_x(f, [\beta], j)$ is zero for any $\mathbb{Z}$-orbit $[\beta]$ with $b(\beta) = 0$ of any multiplicity $j \in \mathbb{N}$.*

Proof. Let $f \in K(x)$ be decomposed into the form (3). If the discrete residue of f at any $\mathbb{Z}$-orbit of any multiplicity is zero, then Lemma 2.4 implies that for all i, j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_i$, the sum

$$\sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x-(\beta_i+\ell))^j} = \Delta_x(g_{i,j}) \quad \text{for some } g_{i,j} \in \overline{K}(x).$$

Since any polynomial is rational summable, there exists $\tilde{p} \in K[x]$ such that $p = \Delta_x(\tilde{p})$. So $f = \Delta_x(\tilde{p} + g)$, where $g = \sum_{i=1}^{m} \sum_{j=1}^{n_i} g_{i,j}$. Arguing as in Proposition 2.2, one sees that for each irreducible factor

p of b, the sum in f is a symmetric function of those β_i's that are roots of p. From this one concludes that the sum is in $K(x)$ and that f is rational summable in $K(x)$.

Suppose that f is rational summable in $K(x)$, i.e., $f = \Delta_x(g)$ for some $g \in K(x)$. Over the field $\overline{K}$, we decompose g into the form (3). For all i, j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_i$, the linearity of Δ_x implies that

$$\Delta_x\left(\sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x-(\beta_i+\ell))^j}\right) = \sum_{\ell=0}^{d_{i,j}+1} \frac{\tilde{\alpha}_{i,j,\ell}}{(x-(\tilde{\beta}_i+\ell))^j},$$

where $\tilde{\beta}_i = \beta_i - 1$, $\tilde{\alpha}_{i,j,0} = \alpha_{i,j,0}$, $\tilde{\alpha}_{i,j,d_{i,j}+1} = -\alpha_{i,j,d_{i,j}}$, and $\tilde{\alpha}_{i,j,\ell} = \alpha_{i,j,\ell} - \alpha_{i,j,\ell-1}$ for $1 \leqslant \ell \leqslant d_{i,j}$. Then the residue $\mathrm{dres}_x(f, [\tilde{\beta}_i], j) = \sum_{\ell=0}^{d_{i,j}} \tilde{\alpha}_{i,j,\ell} = 0$ for all i, j. This completes the proof. □

Remark 2.6. Proposition 2.5 is also known in literature (see [38, Theorem 10] or [10, Corollary 1]). We have recast the known proofs in our terms to show the relevance of discrete residues.

2.3. *q*-discrete residues

Given a rational function, the q-analogue of Abramov's algorithm in [4] can decide whether it is rational q-summable or not. Here, we present a q-analogue of Proposition 2.5 in terms of a q-discrete analogue of residues. To this end, we first recall some terminology from [2–4].

For an element $\alpha \in \overline{K}$, we call the subset $\{\alpha \cdot q^i \mid i \in \mathbb{Z}\}$ of $\overline{K}$ the *$q^{\mathbb{Z}}$-orbit* of α in $\overline{K}$, denoted by $[\alpha]_q$. For a polynomial $b \in K[x] \setminus K$, the value

$$\max\left\{i \in \mathbb{Z} \;\middle|\; \exists \text{ nonzero } \alpha, \beta \in \overline{K} \text{ such that } \alpha = q^i \cdot \beta \text{ and } b(\alpha) = b(\beta) = 0\right\}$$

is called the *q-dispersion* of b with respect to x, denoted by $\mathrm{qdisp}_x(b)$. For $b = \lambda x^n$ with $\lambda \in K$ and $n \in \mathbb{N} \setminus \{0\}$, we define $\mathrm{qdisp}_x(b) = +\infty$. The polynomial b is said to be *q-shift-free* with respect to x if $\mathrm{qdisp}_x(b) = 0$. Let $f = a/b \in K(x)$ be such that $a, b \in K[x]$ and $\gcd(a, b) = 1$. Over the field $\overline{K}$, f can be uniquely decomposed into the form

$$f = c + xp_1 + \frac{p_2}{x^s} + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x - q^\ell \cdot \beta_i)^j}, \tag{4}$$

where $c \in K$, $p_1, p_2 \in K[x]$, $m, n_i \in \mathbb{N}$ are nonzero, $s, d_{i,j} \in \mathbb{N}$, $\alpha_{i,j,\ell}, \beta_i \in \overline{K}$, and β_i's are nonzero and in distinct $q^{\mathbb{Z}}$-orbits.

Definition 2.7 *(q-discrete residue).* Let $f \in K(x)$ be of the form (4). The sum $\sum_{\ell=0}^{d_{i,j}} q^{-\ell\cdot j}\alpha_{i,j,\ell}$ is called the *q-discrete residue* of f at the $q^{\mathbb{Z}}$-orbit $[\beta_i]_q$ of multiplicity j (with respect to x), denoted by $\mathrm{qres}_x(f, [\beta_i]_q, j)$. In addition, we call the constant c the *q-discrete residue* of f at infinity, denoted by $\mathrm{qres}_x(f, \infty)$.

We summarize some basic facts concerning rational q-summability in the next lemma. For a detailed proof, one can see [4, Section 3].

Lemma 2.8. *Let $p, p_1, p_2 \in K[x]$, $c \in K$, and $s \in \mathbb{N} \setminus \{0\}$ be as in (4). Then*

1. $\deg_x(\Delta_{q,x}(p)) = \deg_x(p)$.
2. *If c is nonzero, then c is not rational q-summable in $K(x)$.*
3. *$f = xp_1 + p_2/x^s$ is rational q-summable in $K(x)$.*

The following lemma is a q-analogue of Lemma 2.4 and its proof proceeds in a similar way.

Lemma 2.9. *Let $f=\sum_{\ell=0}^{d}\alpha_\ell/(x-q^\ell\cdot\beta)^s$ be such that $d,s\in\mathbb{N}$, $\alpha_\ell,\beta\in\overline{K}$, and β is nonzero. Then f is rational q-summable in $\overline{K}(x)$ if and only if the sum $\sum_{\ell=0}^{d}q^{-\ell\cdot s}\alpha_\ell$ is zero, that is, if and only if $\operatorname{qres}_x(f,[\beta]_q,s)=0$.*

Proof. Suppose that the sum $\sum_{\ell=0}^{d}q^{-\ell\cdot s}\alpha_\ell$ is zero. We show that f is rational q-summable in $\overline{K}(x)$. To this end, we proceed by induction on d. In the base case when $d=0$, f is clearly rational q-summable since $f=0$. Suppose that the assertion holds for $d=m$ with $m\geqslant 0$. Note that

$$\frac{\alpha_{m+1}}{(x-q^{m+1}\beta)^s}=\Delta_{q,x}\left(-\frac{\alpha_{m+1}}{(x-q^{m+1}\beta)^s}\right)+\frac{q^{-s}\alpha_{m+1}}{(x-q^m\beta)^s}.$$

This implies that

$$\sum_{\ell=0}^{m+1}\frac{\alpha_\ell}{(x-q^\ell\beta)^s}=\Delta_{q,x}\left(-\frac{\alpha_{m+1}}{(x-q^{m+1}\beta)^s}\right)+\sum_{\ell=0}^{m}\frac{\tilde{\alpha}_\ell}{(x-q^\ell\beta)^s},$$

where $\tilde{\alpha}_\ell=\alpha_\ell$ if $0\leqslant\ell\leqslant m-1$ and $\tilde{\alpha}_m=q^{-s}\alpha_{m+1}+\alpha_m$. From the definition and assumption on the α_ℓ's, the sum $\sum_{\ell=0}^{m}q^{-\ell\cdot s}\tilde{\alpha}_\ell$ is zero. The induction hypothesis then implies that there exists $\tilde{g}\in\overline{K}(x)$ such that

$$\sum_{\ell=0}^{m}\frac{\tilde{\alpha}_\ell}{(x-q^\ell\beta)^s}=\Delta_{q,x}(\tilde{g}).$$

So $f=\Delta_{q,x}(g)$ with $g=\tilde{g}-\alpha_{m+1}/(x-q^{m+1}\beta)^s\in\overline{K}(x)$. For the opposite implication, we assume to the contrary that the sum $\sum_{\ell=0}^{d}q^{-\ell\cdot s}\alpha_\ell$ is nonzero. Without loss of generality, we can assume that $\alpha_0\neq 0$. Write $\alpha_0=\bar{\alpha}_0+\tilde{\alpha}_0$ such that $\tilde{\alpha}_0+\sum_{\ell=1}^{d}q^{-\ell\cdot s}\alpha_\ell=0$. Since $\sum_{\ell=0}^{d}q^{-\ell\cdot s}\alpha_\ell\neq 0$, $\bar{\alpha}_0\neq 0$. By the assertion shown above, there exists $\tilde{g}\in\overline{K}(x)$ such that

$$f=\frac{\bar{\alpha}_0}{(x-\beta)^s}+\Delta_{q,x}(\tilde{g}).$$

Since $\operatorname{qdisp}_x((x-\beta)^s)=0$ and $\bar{\alpha}_0\neq 0$, $\bar{\alpha}_0/(x-\beta)^s$ is not rational summable by [33, Lemma 6.3]. Then f is not rational q-summable in $\overline{K}(x)$. This completes the proof. □

Proposition 2.10. *Let $f=a/b\in K(x)$ be such that $a,b\in K[x]$ and $\gcd(a,b)=1$. Then f is rational q-summable in $K(x)$ if and only if the q-discrete residues $\operatorname{qres}_x(f,\infty)$ and $\operatorname{qres}_x(f,[\beta]_q,j)$ are all zero for any $q^{\mathbb{Z}}$-orbit $[\beta]_q$ with $\beta\neq 0$ and $b(\beta)=0$ of any multiplicity $j\in\mathbb{N}$.*

Proof. Let $f\in K(x)$ be decomposed into the form (4). If the residue of f at any $q^{\mathbb{Z}}$-orbit $[\beta]_q$, $\beta\neq 0$, of any multiplicity is zero, then Lemma 2.9 implies that for all i, j with $1\leqslant i\leqslant m$ and $1\leqslant j\leqslant n_i$, the sum

$$\sum_{\ell=0}^{d_{i,j}}\frac{\alpha_{i,j,\ell}}{(x-q^\ell\beta_i)^j}=\Delta_{q,x}(g_{i,j})\quad\text{for some } g_{i,j}\in\overline{K}(x).$$

Since the rational function $xp_1+\frac{p_2}{x^s}$ in (4) is rational q-summable by Lemma 2.8, there exists $u\in K(x)$ such that $xp_1+p_2/x^s=\Delta_{q,x}(u)$. So $f=\Delta_{q,x}(u+g)$, where $g=\sum_{i=1}^{m}\sum_{j=1}^{n_i}g_{i,j}$. As in Proposition 2.5, we see that $g\in K(x)$ and therefore that f is rational q-summable in $K(x)$.

Suppose that f is rational q-summable in $K(x)$, i.e., $f = \Delta_{q,x}(g)$ for some $g \in K(x)$. Over the field $\overline{K}$, we decompose g into the form (4). For all i, j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n_i$, the linearity of $\Delta_{q,x}$ implies that

$$\Delta_{q,x}\left(\sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(x - q^\ell \beta_i)^j}\right) = \sum_{\ell=0}^{d_{i,j}+1} \frac{\tilde{\alpha}_{i,j,\ell}}{(x - q^\ell \tilde{\beta}_i)^j},$$

where $\tilde{\beta}_i = q^{-1}\beta_i$, $\tilde{\alpha}_{i,j,0} = q^{-j}\alpha_{i,j,0}$, $\tilde{\alpha}_{i,j,d_{i,j}+1} = -\alpha_{i,j,d_{i,j}}$, and $\tilde{\alpha}_{i,j,\ell} = q^{-j}\alpha_{i,j,\ell} - \alpha_{i,j,\ell-1}$ for $1 \leqslant \ell \leqslant d_{i,j}$. Then the residue $\operatorname{qres}_x(f, [\tilde{\beta}_i]_q, j) = \sum_{\ell=0}^{d_{i,j}} q^{-\ell \cdot j}\tilde{\alpha}_{i,j,\ell} = 0$ for all i, j. Since $\Delta_{q,x}(c) = 0$ for any constant $c \in k$, the residue of f at infinity is zero. This completes the proof. □

2.4. Residual forms

In terms of residues, we will present a normal form of a rational function in the quotient space $K(x)/\partial_x(K(x))$ with $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$. Let $f \in K(x)$. If f is of the form (2), then we can reduce it to

$$f = D_x(g) + r, \quad \text{where } r = \sum_{i=1}^{m} \frac{\operatorname{cres}_x(f, \beta_i)}{x - \beta_i}.$$

Note that r actually lies in $K(x)$. We call such an r the *residual form* of f with respect to D_x. Similarly, residual forms with respect to Δ_x and $\Delta_{q,x}$ are respectively

$$r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\operatorname{dres}_x(f, [\beta_i], j)}{(x - \beta_i)^j}, \quad \text{where } \beta_i\text{'s in distinct } \mathbb{Z}\text{-orbits}$$

and

$$r = c + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\operatorname{qres}_x(f, [\beta_i]_q, j)}{(x - \beta_i)^j}, \quad \text{where } c \in K \text{ and } \beta_i\text{'s in distinct } q^{\mathbb{Z}}\text{-orbits}.$$

Such a residual form for a rational function is unique up to taking a different representative from orbits. One can compute residual forms without introducing algebraic extensions of K by algorithms in [35,41,36,42–44,4].

3. Algebraic functions

As early as 1827, Abel already observed that an algebraic function satisfies a linear differential equation with polynomial coefficients [1, p. 287]. The annihilating differential equations are important in the study of algebraic functions and their series expansions [22,19,20]. Algorithms for constructing differential annihilators for algebraic functions have been developed in [21,34,23,39,13]. It is not true that any algebraic function satisfies a linear or a q-linear recurrence. In this section we characterize those algebraic functions that satisfy such equations and prove a few lemmas concerning algebraic solutions of first order linear and q-linear recurrences. In the next section, we will see how this restriction on algebraic solutions of such recurrences is responsible for the essential difference between the continuous problems and the (q-)discrete ones.

Let k be an algebraically closed field of characteristic zero. Let $q \in k$ be such that $q^i \neq 1$ for any $i \in \mathbb{Z} \setminus \{0\}$. Let $k(t)$ be the field of all rational functions in t over k. On the field $k(t)$, we let D_t, S_t, and Q_t denote the derivation, shift operator, and q-shift operator with respect to t, respectively. Let

$k(t)\langle D_t\rangle$ (resp. $k(t)\langle S_t\rangle$, $k(t)\langle Q_t\rangle$) denote the ring of linear differential (resp. recurrence, q-recurrence) operators over $k(t)$. We recall the following fact for reference later. One can find its proof in [34, p. 339] or [22, p. 267].

Proposition 3.1. *Let $\alpha(t)$ be an element of the algebraic closure of $k(t)$. Then there exists a nonzero operator $L(t, D_t) \in k(t)\langle D_t\rangle$ such that $L(\alpha) = 0$.*

As mentioned above, the situation is different if we consider the linear (q-)recurrence equations for algebraic functions and the following results show that requiring an algebraic function f to satisfy such a recurrence equation severely restricts f.

Proposition 3.2. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero operator $L(t, S_t) \in k(t)\langle S_t\rangle$ such that $L(\alpha) = 0$, then $\alpha \in k(t)$.*

Proposition 3.3. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero operator $L(t, Q_t) \in k(t)\langle Q_t\rangle$ such that $L(\alpha) = 0$, then $\alpha \in k(t^{1/n})$ for some positive integer n.*

We have included complete proofs (and references to other proofs) of these results in Appendix A.

In the next section, algebraic functions will appear as residues of bivariate rational functions and these functions will satisfy certain first order linear (q-)recurrence relations. The following lemmas characterize the form of these functions. Although these characterizations can be derived from Propositions 3.2 and 3.3, we will give more elementary proofs. Abusing notation, we let S_t and Q_t denote arbitrary extensions of S_t and Q_t to automorphisms of $\overline{k(t)}$, the algebraic closure of $k(t)$.

Lemma 3.4. *Let n be a positive integer.*

(i) *If $f \in \overline{k(t)}$ and $S_t^n(f) = f$, then $f \in k$.*
(ii) *If $f \in \overline{k(t)}$ and $Q_t^n(f) = f$, then $f \in k$.*
(iii) *If $f \in \overline{k(t)}$ and $D_t(f) = 0$, then $f \in k$.*

Proof. (i). We begin by showing that if $f \in k(t)$ and $S_t^n(f) = f$ then $f \in k$. If $f \notin k$, then there exists an element $a \in k$ such that a is a pole or zero of f. In this case the infinite set $\{a + in \mid i \in \mathbb{Z}\}$ will also consist of poles or zeroes, an impossibility since f is a rational function. Now assume that $f \in \overline{k(t)}$ and $S_t^n(f) = f$. Let $Y^\lambda + a_{\lambda-1}Y^{\lambda-1} + \cdots + a_0$ be the minimal polynomial of f over $k(t)$. We then have that $Y^\lambda + S_t^n(a_{\lambda-1})Y^{\lambda-1} + \cdots + S_t^n(a_0)$ is also the minimal polynomial of $f(t) = S_t^n(f(t))$. Therefore $S_t^n(a_i) = a_i$ for all $i = \lambda - 1, \ldots, 0$. This implies that the $a_i \in k$. Since k is algebraically closed, $f \in k$.

(ii). We again begin by showing that if $f \in k(t)$ and $Q_t^n(f) = f$ then $f \in k$. Assume $f \notin k$ and let $a \in k$ be a nonzero pole or zero of f. We then have that the set $\{aq^{in} \mid i \in \mathbb{Z}\}$ consists of poles or zeroes. Since q is not a root of unit, this set is infinite and we get a contradiction as before. Therefore, $f = ct^m$ for some $m \in \mathbb{Z}$. Since $f(q^n t) = f(t)$, we have $q^{nm} = 1$, a contradiction. Therefore $f \in k$. Now assume that $f \in \overline{k(t)}$ and $Q_t^n(f) = f$. An argument similar to that in 1. shows that 2. holds.

(iii). This assertion follows from Lemma 3.3.2(i) of [14, Chapter 3] and the assumption that k is algebraically closed. □

Lemma 3.5. *Let $E \subset F$ be fields of characteristic zero with F algebraic over E. Let σ be an automorphism of F such that $\sigma(E) \subset E$ and let δ be a derivation of F such that $\delta(E) \subset E$. If $\delta\sigma(f) = \sigma\delta(f)$ for all $f \in E$, then $\delta\sigma(f) = \sigma\delta(f)$ for all $f \in F$.*

Proof. One can verify that $\sigma^{-1}\delta\sigma$ is a derivation on F such that $\sigma^{-1}\delta\sigma(E) \subset E$. Therefore $\sigma^{-1}\delta\sigma - \delta$ is a derivation on f that is zero on E. From the uniqueness of extensions of derivations to algebraic extensions, we have that $\sigma^{-1}\delta\sigma - \delta$ is zero on F, which yields the result. □

Lemma 3.6. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero $n \in \mathbb{N}$ such that $S_t^n(\alpha) = q^m\alpha$ for some $m \in \mathbb{Z}$, then $m = 0$ and $\alpha(t) \in k$.*

Proof. Let $\delta = D_t$. Lemma 3.5 implies that $S_t^n\delta = \delta S_t^n$ on $\overline{k(t)}$. Therefore, $S_t^n(\delta\alpha) = q^m\delta\alpha$. One see that this implies that $S_t^n(\delta\alpha/\alpha) = \delta\alpha/\alpha$, so by Lemma 3.4 $\delta\alpha = c\alpha$ for some $c \in k$. Assume that $\alpha \notin k$ and therefore that $\delta\alpha \neq 0$ and $c \neq 0$. Let $P(Y) = Y^\lambda + a_{\lambda-1}Y^{\lambda-1} + \cdots + a_0$ be the minimal polynomial of α over $k(t)$. Applying δ to $P(\alpha)$, one sees that

$$Y^\lambda + \frac{\delta a_{\lambda-1} + (\lambda-1)c}{\lambda c}Y^{\lambda-1} + \cdots + \frac{\delta a_0}{\lambda c}$$

is also the minimal polynomial of α over $k(t)$. Therefore

$$\frac{\delta a_0}{a_0} = \lambda c.$$

Since $a_0 \in k(t)$, we may write $a_0 = d\prod(t-e_i)^{\mu_i}$, where $d, e_i \in k$, $\mu_i \in \mathbb{Z}$. Therefore

$$\sum \frac{\mu_i}{t-e_i} = \lambda c$$

contradicting the uniqueness of partial fraction decomposition. This contradiction implies that $\alpha \in k$. From the equation $S_t^n(\alpha) = q^m\alpha$ we get $q^m = 1$. Therefore $m = 0$ since q is not root of unity. □

Lemma 3.7. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero $n \in \mathbb{Z}$ such that $S_t^n(\alpha) - \alpha = m$ for some $m \in \mathbb{Z}$, then $\alpha(t) = \frac{m}{n}t + c$ for some $c \in k$.*

Proof. Let $\beta(t) = \frac{m}{n}t$. Since $S_t^n(\beta) - \beta = m$, we have that $S_t^n(\alpha - \beta) - (\alpha - \beta) = 0$. Therefore Lemma 3.4 implies that $\alpha = \beta + c = \frac{m}{n}t + c$ for some $c \in k$. □

Lemma 3.8. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero $n \in \mathbb{Z}$ such that $Q_t^n(\alpha) - \alpha = m$ for some $m \in \mathbb{Z}$, then $m = 0$ and $\alpha(t) \in k$.*

Proof. Let $\delta = tD_t$. One has that $\delta Q_t = Q_t\delta$ on $k(t)$ so Lemma 3.5 implies that $\delta Q_t = Q_t\delta$ on $\overline{k(t)}$. We then also have $\delta Q_t^n = Q_t^n\delta$ on $\overline{k(t)}$ so $Q_t^n(\delta\alpha) - \delta\alpha = 0$. Lemma 3.4 implies $\delta\alpha \in k$. Suppose that $\delta\alpha = c$ for $c \in k$. Then $D_t(\alpha) = c/t$. If $\mathrm{Tr} : k(t)(\alpha) \to k(t)$ is the trace mapping, then $D_t(\mathrm{Tr}(\alpha)) = \lambda c/t$ for some nonzero $\lambda \in \mathbb{N}$. By Proposition 2.2, we have $\lambda c = 0$ and then $c = 0$. Now $\alpha \in k$ follows from the third assertion of Lemma 3.4. □

Lemma 3.9. *Let $\alpha(t)$ be an element in the algebraic closure of $k(t)$. If there exists a nonzero $n \in \mathbb{Z}$ such that $Q_t^n(\alpha) = q^m\alpha$ for some $m \in \mathbb{Z}$, then $\alpha(t) = ct^{\frac{m}{n}}$ for some $c \in k$.*

Proof. Let $\beta(t) = t^{\frac{m}{n}}$. We then have that

$$Q_t^n\left(\frac{\alpha}{\beta}\right) = \frac{\alpha}{\beta}$$

so $\alpha/\beta = c \in k$ by Lemma 3.4, that is, $\alpha = ct^{\frac{m}{n}}$. □

Table 1
Nine different types of telescoping equations.

(L, ∂_x)	D_x	Δ_x	$\Delta_{q,x}$
$k(t)\langle D_t\rangle$	$L(t, D_t)(f) = D_x(g)$	$\underline{L(t, D_t)(f) = \Delta_x(g)}$	$\underline{L(t, D_t)(f) = \Delta_{q,x}(g)}$
$k(t)\langle S_t\rangle$	$\underline{L(t, S_t)(f) = D_x(g)}$	$L(t, S_t)(f) = \Delta_x(g)$	$\underline{L(t, S_t)(f) = \Delta_{q,x}(g)}$
$k(t)\langle Q_t\rangle$	$\underline{L(t, Q_t)(f) = D_x(g)}$	$\underline{L(t, Q_t)(f) = \Delta_x(g)}$	$L(t, Q_t)(f) = \Delta_{q,x}(g)$

4. Telescopers

In Section 2, we see that nonzero residues are the obstruction for a rational function to being rational integrable (resp. summable, q-summable). In this section, we consider whether we can use a linear operator, a so-called *telescoper*, to remove this obstruction if an extra parameter is available. The importance of telescopers in the study of special functions and combinatorial identities have been shown in the work by Zeilberger and his collaborators [55,9,53,52,54].

Let $k(t, x)$ be the field of rational functions in t and x over k. On the field $k(t, x)$, we have derivations D_t, D_x, shift operators S_t, S_x, and q-shift operators Q_t, Q_x. The linear operators used below will be in the ring $k(t)\langle D_t\rangle$, $k(t)\langle S_t\rangle$, or $k(t)\langle Q_t\rangle$. For a rational function $f \in k(t, x)$, we wish to solve the Existence Problem for Telescopers stated in the Introduction, that is, we want to decide the existence of linear operators $L(t, \partial_t)$ with $\partial_t \in \{D_t, S_t, Q_t\}$ such that

$$L(t, \partial_t)(f) = \partial_x(g) \tag{5}$$

for some $g \in k(t, x)$ and $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$. According to the different choices of L and ∂_x, we have nine types of telescopers in general, see Table 1.

The existence problem of telescopers is related to the termination of Zeilberger-style algorithms and has been studied in [8,6,17,15] but, to our knowledge, our results concerning telescopers of the six types underlined in the above table are new. In this section, we will present a unified way to solve this problem for rational functions by using the knowledge in the previous sections. Before the investigation of the existence of telescopers, we first present some preparatory lemmas for later use.

Definition 4.1. Let $\sim$ be an equivalence relation on a set R and $\sigma : R \to R$ be a bijection. The relation $\sim$ is said to be σ-*compatible* if

$$\sigma(r_1) \sim \sigma(r_2) \quad \Leftrightarrow \quad r_1 \sim r_2 \quad \text{for all } r_1, r_2 \in R.$$

If the equivalence relation $\sim$ is compatible with a bijection σ on R, then a bijection on the quotient set $R/\sim$ can be naturally induced by σ, for which we still use the name σ. We denote by $[t]$ the equivalence class of t in $R/\sim$.

Proposition 4.2. *Let $\sigma : R \to R$ be a bijection and $\sim$ be a σ-compatible equivalence relation on the set R. Let $T = \{[t_1], \ldots, [t_n]\} \subset R/\sim$. If for any $i \in \{1, \ldots, n\}$, there exists nonzero $m_i \in \mathbb{N}$ such that $\sigma^{m_i}([t_i]) \in T$, then there exists nonzero $m \in \mathbb{N}$ such that $\sigma^m([t_i]) = [t_i]$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let $\tilde{m}$ be the least common multiple of m_i's. Then $\sigma^{\tilde{m}}$ is a permutation on the finite set T. Since any permutation on a finite set is idempotent, there exists an $s \in \mathbb{N}$ such that $\sigma^{\tilde{m}s}$ is an identity on T. Taking $m = \tilde{m}s$ completes the proof. □

We will specialize Proposition 4.2 to different bijections and equivalence relations. The following examples show how to perform specializations.

Example 4.3. Let R be the algebraic closure of $k(t)$. The equivalence relation $\sim$ on R is defined by $\alpha_1 \sim \alpha_2$ if and only if $\alpha_1 - \alpha_2 \in \mathbb{Z}$. We take the shift mapping $\sigma(\alpha(t)) = \alpha(t + 1)$ as the bijection.

Let $T = \{[\alpha_1], \ldots, [\alpha_n]\}$ be such that for any $i \in \{1, \ldots, n\}$, $\sigma^{m_i}([\alpha_i]) \in T$ for some nonzero $m_i \in \mathbb{N}$. By Proposition 4.2, there exists nonzero $m \in \mathbb{N}$ such that $\sigma^m(\alpha_i) - \alpha_i \in \mathbb{Z}$ for all $i \in \{1, \ldots, n\}$. Applying Lemma 3.7 to α_i yields $\alpha_i = \frac{n_i}{m}t + c_i$ for some $n_i \in \mathbb{Z}$ and $c_i \in k$.

Example 4.4. Let R be the algebraic closure of $k(t)$. The equivalence relation $\sim$ on R is defined by $\alpha_1 \sim \alpha_2$ if and only if $\alpha_1/\alpha_2 \in q^{\mathbb{Z}}$. We take the q-shift mapping $\sigma(\alpha(t)) = \alpha(qt)$ as the bijection. Let $T = \{[\alpha_1]_q, \ldots, [\alpha_n]_q\}$ be such that for any $i \in \{1, \ldots, n\}$, $\sigma^{m_i}([\alpha_i]) \in T$ for some nonzero $m_i \in \mathbb{N}$. By Proposition 4.2, there exists nonzero $m \in \mathbb{N}$ such that $\sigma^m(\alpha_i)/\alpha_i \in q^{\mathbb{Z}}$ for all $i \in \{1, \ldots, n\}$. Applying Lemma 3.9 to α_i yields $\alpha_i = c_i t^{n_i/m}$ for some $n_i \in \mathbb{Z}$ and $c_i \in k$.

4.1. *Existence of telescopers*

The first result about the existence of telescopers was shown by Zeilberger in [55] based on the theory of holonomic D-modules. In the following, we will study the existence problems from the residual point of view. For rational functions, the existence of telescopers is related to the properties of residues and the commutativity between the residue mappings and linear operators.

Starting from the simplest, we consider the telescoping relation $L(t, D_t)(f) = D_x(g)$ for a given rational function $f \in k(t, x)$. Given $\beta \in \overline{k(t)}$, view the residue mapping $\mathrm{cres}_x(\_, \beta)$ as a $\overline{k(t)}$-linear transformation from $\overline{k(t)}(x)$ to $\overline{k(t)}$. For any $\alpha, \beta \in \overline{k(t)}$, we have

$$D_t\left(\frac{\alpha}{x-\beta}\right) = \frac{D_t(\alpha)}{x-\beta} + \frac{\alpha D_t(\beta)}{(x-\beta)^2}.$$

Then $\mathrm{cres}_x(D_t(f), \beta) = D_t(\mathrm{cres}_x(f, \beta))$ for any $f \in \overline{k(t)}(x)$ and $\beta \in \overline{k(t)}$. Assume that $f = a/b$ with $a, b \in k[t, x]$ and $\gcd(a, b) = 1$. Let $\beta_1, \ldots, \beta_m$ be the roots of b in $\overline{k(t)}$. For each root β_i, the continuous residue $\mathrm{cres}_x(f, \beta_i) \in \overline{k(t)}$ is annihilated by a linear differential operator $L_i \in k(t)\langle D_t\rangle$ by Proposition 3.1. Let $L(t, D_t)$ be the least common left multiple (LCLM) of the L_i's. Then we have $L(\mathrm{cres}_x(f, \beta_i)) = \mathrm{cres}_x(L(f), \beta_i) = 0$ for all i with $1 \leqslant i \leqslant m$. So $L(f)$ is rational integrable with respect to x by Proposition 2.2. In summary, we have the following theorem.

Theorem 4.5. *For any $f \in k(t, x)$, there exists a nonzero operator $L \in k(t)\langle D_t\rangle$ such that $L(f) = D_x(g)$ for some $g \in k(t, x)$.*

However, the situation in other cases turns out to be more involved. For the rational function $f = 1/(t^2 + x^2)$, Abramov and Le [37,8] showed that there is no telescoper in $k(t)\langle S_t\rangle$ such that $L(f) = \Delta_x(g)$ for any $g \in k(t, x)$. In other cases, there are two main reasons for non-existence: one is the non-commutativity between linear operators $\partial_t \in \{D_t, S_t, Q_t\}$ and residue mappings, the other is that not all algebraic functions would satisfy linear (q)-recurrence relations. So it is natural that rational functions are of special forms if telescopers exist.

Let $f \in k(t, x)$ and $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$. Then $f = \partial_x(g) + r$ with $g, r \in k(t, x)$ and r being the residual form of f with respect to ∂_x (see Section 2.4). Since linear operators $L(t, \partial_t)$ with $\partial_t \in \{D_t, S_t, Q_t\}$ commute with the linear operator $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$, a rational function has a telescoper if and only if its residual form does. From now on, we always assume that the given rational function is in its residual form. We will also use the fact [8, Lemma 1] that the sum $f_1 + f_2$ has a telescoper if both f_1 and f_2 do. To be more precise, if L_1, L_2 are telescopers for f_1, f_2, respectively, then the LCLM of L_1, L_2 is a telescoper for $f_1 + f_2$.

4.1.1. *Telescopers with respect to D_x*

Let $f \in k(t, x)$ be a residual form, that is,

$$f = \sum_{i=1}^{m} \frac{\alpha_i}{x - \beta_i}, \quad \text{where } \alpha_i, \beta_i \in \overline{k(t)} \text{ and the } \beta_i \text{ are pairwise distinct.} \tag{6}$$

Theorem 4.6. *Let $f \in k(t,x)$ be as in (6). Then f has a telescoper L in $k(t)\langle S_t\rangle$ such that $L(t,S_t)(f) = D_x(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle S_t\rangle$ such that $L(t,S_t)(f) = D_x(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell S_t^\ell$ with $e_\ell \in k(t)$ and $e_\rho = 1$. Then

$$L(f) = \sum_{\ell=0}^{\rho}\sum_{i=1}^{m} \frac{e_\ell S_t^\ell(\alpha_i)}{x - S_t^\ell(\beta_i)}.$$

Assume that ℓ_0 is the first index in $\{0,1,\ldots,\rho\}$ such that $e_{\ell_0} \neq 0$. Since $L(f)$ is rational integrable in $k(t,x)$ with respect to D_x, all residues of $L(f)$ are zero by Proposition 2.2. In particular, the set $T = \{S_t^{\ell_0}(\beta_1),\ldots,S_t^{\ell_0}(\beta_m)\}$ satisfies the property that for any $i \in \{1,\ldots,m\}$, there exists nonzero $m_i \in \mathbb{N}$ such that $S_t^{\ell_0+m_i}(\beta_i) \in T$. By taking equality as the equivalence relation and the shift mapping as the bijection in Proposition 4.2, there exists nonzero $m \in \mathbb{N}$ such that $S_t^{\ell_0+m}(\beta_i) = \beta_i$ for all $i \in \{1,\ldots,m\}$. By Lemma 3.4(i) and the assumption that k is algebraically closed, all the β_i are in k.

For the opposite implication, it suffices to show that each fraction $\alpha_i/(x-\beta_i)$ with $\beta_i \in k$ has a telescoper in $k(t)\langle S_t\rangle$. According to the process of partial fraction decomposition, $\alpha_i \in k(t)(\beta_i)$ for any i with $1 \leqslant i \leqslant m$. Then $\alpha_i \in k(t)$, which is annihilated by the operator $L_i = S_t - \alpha_i(t+1)/\alpha_i(t)$. Moreover, $L_i(\alpha_i/(x-\beta_i)) = L_i(\alpha_i)/(x-\beta_i) = 0$. So the LCLM of the L_i's is a telescoper for f. This completes the proof. □

Theorem 4.7. *Let $f \in k(t,x)$ be as in (6). Then f has a telescoper L in $k(t)\langle Q_t\rangle$ such that $L(t,Q_t)(f) = D_x(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. The proof proceeds in a similar way as above replacing S_t by Q_t and Lemma 3.4(i) by Lemma 3.4(ii). □

Example 4.8. Let $f = 1/(x+t)$. Since the root of $x+t$ in $\overline{k(t)}$ is t, which is not in k, f has no telescoper in either $k(t)\langle S_t\rangle$ or $k(t)\langle Q_t\rangle$ with respect to D_x by Theorems 4.6 and 4.7.

4.1.2. Telescopers with respect to Δ_x

Let $f \in k(t,x)$ be of the form

$$f = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x-\beta_i)^j}, \tag{7}$$

where $\alpha_{i,j}, \beta_i \in \overline{k(t)}$, $\alpha_{i,n_i} \neq 0$, and the β_i are in distinct $\mathbb{Z}$-orbits.

Theorem 4.9. *Let $f \in k(t,x)$ be as in (7). Then f has a telescoper L in $k(t)\langle D_t\rangle$ such that $L(t,D_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle D_t\rangle$ such that $L(t,D_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell D_t^\ell$ with $e_\ell \in k(t)$. By induction on ℓ, we get

$$D_t^\ell\left(\frac{\alpha_{i,n_i}}{(x-\beta_i)^{n_i}}\right) = \frac{(n_i)_\ell \alpha_{i,n_i}(D_t(\beta_i))^\ell}{(x-\beta_i)^{n_i+\ell}} + \text{lower terms},$$

where $(n_i)_\ell = n_i(n_i+1)\cdots(n_i+\ell-1)$. Then we have

$$L(f)=\sum_{i=1}^{m}\frac{(n_i)_\rho\alpha_{i,n_i}(D_t(\beta_i))^\rho}{(x-\beta_i)^{n_i+\rho}}+\text{lower terms.}$$

Since $L(f)$ is rational summable with respect to Δ_x and the β_i are in distinct $\mathbb{Z}$-orbits, we get $(n_i)_\rho\alpha_{i,n_i}(D_t(\beta_i))^\rho = 0$ for all $i \in \{1,\dots,m\}$ by Proposition 2.5. Since $\alpha_{i,n_i} \neq 0$ and $(n_i)_\rho > 0$, $D_t(\beta_i) = 0$, which implies that $\beta_i \in k$ by Lemma 3.4(iii).

For the opposite implication, the proof is similar to that of Theorem 4.6. Let $L_{i,j}$ be the operator $D_t - D_t(\alpha_{i,j})/\alpha_{i,j} \in k(t)\langle D_t\rangle$. Then the LCLM of the $L_{i,j}$ is a telescoper for f with respect to Δ_x. □

Example 4.10. Let

$$f=\frac{1}{x^2-t}=\frac{1}{2\sqrt{t}}\left(\frac{1}{x-\sqrt{t}}-\frac{1}{x+\sqrt{t}}\right).$$

Note that f is already in residual form with respect to Δ_x. By Theorem 4.9, there is no linear differential operator $L(t, D_t) \in k(t)\langle D_t\rangle$ and $g \in k(t,x)$ such that $L(t, D_t) f = \Delta_x(g)$. Furthermore, Proposition 3.1 in [33] and the descent argument similar to that given in the proof of Corollary 3.2 of [33] (or Section 1.2.1 of [24]) implies that the sum

$$F(t,x)=\sum_{i=1}^{x-1}\frac{1}{i^2-t}\quad \bigl(\text{satisfying } S_x(F)-F=f\bigr)$$

satisfies no polynomial differential equation $P(t, x, F, D_tF, D_t^2F, \dots) = 0$.

The following theorem is the same as in [8, Theorem 1]. We give an alternative proof using the knowledge developed in the previous sections.

Theorem 4.11. *Let $f \in k(t,x)$ be as in (7). Then f has a telescoper L in $k(t)\langle S_t\rangle$ such that $L(t, S_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$ if and only if all the $\beta_i = r_i t + c_i$ with $r_i \in \mathbb{Q}$ and $c_i \in k$.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle S_t\rangle$ such that $L(t, S_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell S_t^\ell$ with $e_\ell \in k(t)$ and $e_0 \neq 0$. For any $\lambda \in \{1,\dots,m\}$, we consider the rational function

$$f_\lambda=\sum_{i=1}^{m}\frac{\alpha_{i,n_\lambda}}{(x-\beta_i)^{n_\lambda}},\quad \text{where } \alpha_{\lambda,n_\lambda}\neq 0 \text{ by assumption.}$$

Without loss of generality, we may assume that the other α_{i,n_λ} with $i \neq \lambda$ are also nonzero. Since the shift operators S_t, S_x preserve the multiplicity, we have $L(f_\lambda) = \Delta_x(g_\lambda)$ for some $g_\lambda \in k(t,x)$. By Proposition 2.5, all the residues of $L(f_\lambda)$ are zero. We now use the notation and analysis of Example 4.3. We see that the set $T = \{[\beta_1],\dots,[\beta_m]\}$ satisfies the property that for any $i \in \{1,\dots,m\}$, there exists a nonzero m_i such that $S_t^{m_i}([\beta_i]) \in T$. As in Example 4.3, we conclude that $\beta_i = \frac{p_i}{m}t + c_i$ with $p_i, m \in \mathbb{Z}$ and $c_i \in k$.

The opposite implication follows from the fact that the linear operator

$$L_{i,j}=\alpha_{i,j}(t)S_t^m-\alpha_{i,j}(t+m)$$

is a telescoper for the fraction $f_{i,j} = \alpha_{i,j}/(x - (\frac{p_i}{m}t + c_i))^j$ with respect to Δ_x since $\mathrm{dres}(L_{i,j}(f_{i,j}), [\frac{p_i}{m}t + c_i], j) = 0$. Then the LCLM of the $L_{i,j}$ is a telescoper for f with respect to Δ_x. □

Theorem 4.12. *Let $f \in k(t,x)$ be as in (7). Then f has a telescoper L in $k(t)\langle Q_t\rangle$ such that $L(t, Q_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle Q_t\rangle$ such that $L(t, Q_t)(f) = \Delta_x(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell Q_t^\ell$ with $e_\ell \in k(t)$ and $e_0 \neq 0$. For any $\lambda \in \{1, \ldots, m\}$, we consider the rational function

$$f_\lambda = \sum_{i=1}^{m} \frac{\alpha_{i,n_\lambda}}{(x - \beta_i)^{n_\lambda}}, \quad \text{where } \alpha_{\lambda,n_\lambda} \neq 0 \text{ by assumption.}$$

Without loss of generality, we may assume that the other α_{i,n_λ} with $i \neq \lambda$ are also nonzero. Since the operators Q_t, S_x preserve the multiplicity, we have $L(f_\lambda) = \Delta_x(g_\lambda)$ for some $g_\lambda \in k(t,x)$. By Proposition 2.5, all the residues of $L(f_\lambda)$ are zero. We shall again use the reasoning and notation in Example 4.3 where [ ] is an equivalence class of the equivalence relation that $\alpha_1 \sim \alpha_2$ in $\overline{k(t)}$ if $\alpha_1 - \alpha_2 \in \mathbb{Z}$. In particular, the set $T = \{[\beta_1], \ldots, [\beta_m]\}$ satisfies the property that for any $i \in \{1, \ldots, m\}$, there exists a nonzero m_i such that $Q_t^{m_i}([\beta_i]) \in T$. Taking the shift mapping Q_t as the bijection, Proposition 4.2 and Lemma 3.8 imply that $\beta_i \in k$ for all i with $1 \leqslant i \leqslant m$.

The opposite implication follows from the fact that the linear operator

$$L_{i,j} = \alpha_{i,j}(t) Q_t - \alpha_{i,j}(qt)$$

is a telescoper for the fraction $f_{i,j} = \alpha_{i,j}/(x - \beta_i)^j$ with respect to Δ_x since $\mathrm{dres}(L_{i,j}(f_{i,j}), [\beta_i], j) = 0$. Then the LCLM of the $L_{i,j}$ is a telescoper for f with respect to Δ_x. □

4.1.3. Telescopers with respect to $\Delta_{q,x}$

Let $f \in k(t,x)$ be of the form

$$f = c + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x - \beta_i)^j}, \tag{8}$$

where $c \in k(t)$, $\alpha_{i,j}, \beta_i \in \overline{k(t)}$, $\alpha_{i,n_i} \neq 0$, and the β_i are in distinct $q^{\mathbb{Z}}$-orbits.

Theorem 4.13. *Let $f \in k(t,x)$ be as in (8). Then f has a telescoper L in $k(t)\langle D_t\rangle$ such that $L(t, D_t)(f) = \Delta_{q,x}(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. The proof proceeds in the same way as that in Theorem 4.9. □

Theorem 4.14. *Let $f \in k(t,x)$ be as in (8). Then f has a telescoper L in $k(t)\langle S_t\rangle$ such that $L(t, S_t)(f) = \Delta_{q,x}(g)$ for some $g \in k(t,x)$ if and only if all the β_i are in k.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle S_t\rangle$ such that $L(t, S_t)(f) = \Delta_{q,x}(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell S_t^\ell$ with $e_\ell \in k(t)$ and $e_0 \neq 0$. For any $\lambda \in \{1, \ldots, m\}$, we consider the rational function

$$f_\lambda = \sum_{i=1}^{m} \frac{\alpha_{i,n_\lambda}}{(x - \beta_i)^{n_\lambda}}, \quad \text{where } \alpha_{\lambda,n_\lambda} \neq 0 \text{ by assumption.}$$

Without loss of generality, we may assume that the other α_{i,n_λ} with $i \neq \lambda$ are also nonzero. Since the operators S_t, Q_x preserve the multiplicity, we have $L(f_\lambda) = \Delta_{q,x}(g_\lambda)$ for some $g_\lambda \in k(t,x)$. By Proposition 2.10, all the residues of $L(f_\lambda)$ are zero. We now use the reasoning and notation in Example 4.4. In particular, the set $T = \{[\beta_1]_q, \ldots, [\beta_m]_q\}$ satisfies that for any $i \in \{1, \ldots, m\}$, there exists a nonzero m_i such that $S_t^{m_i}([\beta_i]_q) \in T$. Taking the shift mapping S_t as bijection, Proposition 4.2 and Lemma 3.6 imply that $\beta_i \in k$ for all i with $1 \leqslant i \leqslant m$.

The opposite implication follows from the fact that $c(t)$ is annihilated by the operator $L_0 = c(t)S_t - c(t+1)$ and the linear operator

$$L_{i,j} = \alpha_{i,j}(t)S_t - \alpha_{i,j}(t+1)$$

is a telescoper for the fraction $f_{i,j} = \alpha_{i,j}/(x-\beta_i)^j$ with respect to $\Delta_{q,x}$ since $\operatorname{dres}(L_{i,j}(f_{i,j}), [\beta_i]_q, j) = 0$. Then the LCLM of the L_0 and $L_{i,j}$ is a telescoper for f with respect to $\Delta_{q,x}$. □

The following theorem is a q-analogue of Theorem 4.11, which has also been shown in [37, Theorem 1].

Theorem 4.15. *Let $f \in k(t,x)$ be as in (8). Then f has a telescoper L in $k(t)\langle Q_t\rangle$ such that $L(t,Q_t)(f) = \Delta_{q,x}(g)$ for some $g \in k(t,x)$ if and only if all the $\beta_i = c_i t^{r_i}$ with $r_i \in \mathbb{Q}$ and $c_i \in k$.*

Proof. Suppose that there exists a nonzero $L \in k(t)\langle Q_t\rangle$ such that $L(t,Q_t)(f) = \Delta_{q,x}(g)$ for some $g \in k(t,x)$. Write $L = \sum_{\ell=0}^{\rho} e_\ell Q_t^\ell$ with $e_\ell \in k(t)$ and $e_0 \neq 0$. For any $\lambda \in \{1, \ldots, m\}$, we consider the rational function

$$f_\lambda = \sum_{i=1}^{m} \frac{\alpha_{i,n_\lambda}}{(x-\beta_i)^{n_\lambda}}, \quad \text{where } \alpha_{\lambda,n_\lambda} \neq 0 \text{ by assumption.}$$

Without loss of generality, we may assume that the other α_{i,n_λ} with $i \neq \lambda$ are also nonzero. Since the q-shift operators Q_t, Q_x preserve the multiplicity, we have $L(f_\lambda) = \Delta_{q,x}(g_\lambda)$ for some $g_\lambda \in k(t,x)$. By Proposition 2.10, all the residues of $L(f_\lambda)$ are zero. In particular, the set $T = \{[\beta_1]_q, \ldots, [\beta_m]_q\}$ satisfies that for any $i \in \{1, \ldots, m\}$, there exists a nonzero m_i such that $Q_t^{m_i}([\beta_i]_q) \in T$. By the analysis in Example 4.4, we conclude that $\beta_i = c_i t^{p_i/m}$ with $p_i, m \in \mathbb{Z}$ and $c_i \in k$.

The opposite implication follows from the fact that $c(t)$ is annihilated by the operator $L_0 = cS_t - c(t+1)$ and the linear operator

$$L_{i,j} = \alpha_{i,j}(t)Q_t^m - q^{-jp_i}\alpha_{i,j}\big(q^m t\big)$$

is a telescoper for the fraction $f_{i,j} = \alpha_{i,j}/(x-(c_i t^{p_i/m}))^j$ with respect to $\Delta_{q,x}$ since $\operatorname{qres}(L_{i,j}(f_{i,j}), [c_i t^{p_i/m}]_q, j) = 0$. Then the LCLM of the L_0 and $L_{i,j}$ is a telescoper for f with respect to $\Delta_{q,x}$. □

The necessary and sufficient conditions for the existence of telescopers enable us to decide the termination of the Zeilberger algorithm for rational-function inputs. After reducing the given rational function into a residual form, one can detect the existence by investigating the denominator. For instance, we could check whether the denominator factors into two univariate polynomials respectively in t and x in the case when $\partial_t = D_t$ and $\partial_x = \Delta_x$. Combining the existence criteria with the Zeilberger algorithm yields a complete algorithm for creative telescoping with rational-function inputs.

4.2. Characterization of telescopers

We have shown that telescopers exist for a special class of rational functions. Now, we will characterize the linear differential and (q-)recurrence operators that could be telescopers for rational functions. Using such a characterization, we will give a direct algebraic proof of a theorem of Furstenberg stating that the diagonal of a rational power series in two variables is algebraic [29]. In all of these considerations, residues are still the key.

For a rational function $f \in k(t,x)$, all of the telescopers for f in $k(t)\langle D_t\rangle$ form a left ideal in $k(t)\langle D_t\rangle$, denoted by $\mathcal{T}_f$. Since the ring $k(t)\langle D_t\rangle$ is a left Euclidean domain, the monic telescoper of minimal order generates the left ideal $\mathcal{T}_f$, and we call this generator *the minimal telescoper* for f.

Theorem 4.16. *Let $L(t, D_t)$ be a linear differential operator in $k(t)\langle D_t\rangle$. Then L is a telescoper for some $f \in k(t,x) \setminus D_x(k(t,x))$ such that $L(f) = D_x(g)$ with $g \in k(t,x)$ if and only if $L(y(t)) = 0$ has a nonzero solution algebraic over $k(t)$. Moreover, if L is the minimal telescoper for f, then all solutions of $L(y(t)) = 0$ are algebraic over $k(t)$.*

Proof. Suppose that there exists $f \in k(t,x) \setminus D_x(k(t,x))$ such that $L(f) = D_x(g)$ for some $g \in k(t,x)$. Since f is not rational integrable with respect to x, f has a nonzero residue by Proposition 2.2. Since L is a telescoper for f with respect to D_x, L vanishes at all residues of f. So $L(y(t)) = 0$ has a nonzero algebraic solution in $\overline{k(t)}$ because any residue of a rational function in $k(t,x)$ is algebraic over $k(t)$.

Conversely, if $\alpha \in \overline{k(t)}$ is a nonzero algebraic solution of $L(y(t)) = 0$ with minimal polynomial $P \in k[t,x]$, then L is a telescoper for the rational function $f = xD_x(P)/P$ with respect to D_x.

Let $a/b \in k(t,x)$ be the residual form of f with respect to D_x. All of the residues of a/b are roots of the polynomial $R(t,z) = \text{resultant}_x(b, a - zD_x(b)) \in k(t)[z]$. By the method in [23, Section 2], one can construct the minimal operator L_R in $k(t)\langle D_t\rangle$ such that $L_R(\alpha(t)) = 0$ for all roots of R in $\overline{k(t)}$. Moreover, the solutions space of L_R is spanned by the roots of R. Since L_R vanishes at all residues of f, L_R is a telescoper for f. If L is the minimal telescoper for f, then L divides L_R on the right. Thus, all solutions of $L(y(t)) = 0$ are solutions of $L_R(y(t)) = 0$, and therefore algebraic over $k(t)$. □

The diagonal $\text{diag}(f)$ of a formal power series $f = \sum_{i,j\geqslant 0} f_{i,j}t^i x^j \in k[[t,x]]$ is defined by

$$\text{diag}(f) = \sum_{i\geqslant 0} f_{i,i}t^i \in k[[t]].$$

Using the characterization of telescopers in Theorem 4.16, we now give a proof of a theorem of Furstenberg that the diagonal of a rational power series in two variables is algebraic [29]. For other proofs, see the papers [27,30,32] and Stanley's book [48, Theorem 6.3.3]. Several of these authors use residues in their proofs of Furstenberg's result. The novelty in our proof is the use of minimal telescopers and the property described in Theorem 4.16.

Let $\mathcal{F} = k((x))$ be the quotient field of $k[[x]]$ and $\mathcal{F}[[t]]$ be the formal power series over $\mathcal{F}$. We use the notation $[x^{-1}](a)$ to denote the coefficient of x^{-1} in $a \in \mathcal{F}$. For a formal power series $g = \sum_{i\geqslant 0} a_i(x)t^i \in \mathcal{F}[[t]]$, we define

$$[x^{-1}](g) = \sum_{i\geqslant 0} \big([x^{-1}](a_i)\big)t^i \in k[[t]],$$

and two derivations

$$D_t(g) = \sum_{i\geqslant 0} i a_i(x)t^{i-1}, \qquad D_x(g) = \sum_{i\geqslant 0} D_x(a_i)t^i.$$

The ring $\mathcal{F}[[t]]$ then becomes a $k[t,x]\langle D_t, D_x\rangle$-module. By definition, we have

$$\left[x^{-1}\right]\left(D_t(g)\right) = D_t\left(\left[x^{-1}\right](g)\right) \quad \text{and} \quad \left[x^{-1}\right]\left(t^i(g)\right) = t^i\left(\left[x^{-1}\right](g)\right)$$

for all $i \in \mathbb{N}$. By induction, we have $L([x^{-1}](g)) = [x^{-1}](L(g))$ for all $L \in k[t]\langle D_t\rangle$. Since $[x^{-1}](D_x(a)) = 0$ for any $a \in \mathcal{F}$, we get $[x^{-1}](D_x(g)) = 0$ for any $g \in \mathcal{F}[[t]]$. Let $f = \sum_{i,j \geqslant 0} f_{i,j} t^i x^j$ be a formal power series in $k[[t,x]]$. Then $F = f(x, t/x)/x$ is in $\mathcal{F}[[t]]$. Applying $[x^{-1}]$ to F yields

$$\left[x^{-1}\right](F) = \left[x^{-1}\right]\left(\sum_{i,j\geqslant 0} f_{i,j} x^{i-j-1} t^j\right) = \sum_{j \geqslant 0} f_{j,j} t^j = \operatorname{diag}(f).$$

If $L \in k[t]\langle D_t\rangle$ be such that $L(F) = D_x(G)$ for some $G \in \mathcal{F}[[t]]$, then applying $[x^{-1}]$ to both sides of $L(F) = D_x(G)$ yields $L(\operatorname{diag}(f)) = 0$. In summary, we have the following lemma.

Lemma 4.17. *Let $f \in k[[t,x]]$ and $F = f(x,t/x)/x \in \mathcal{F}[[t]]$. If $L \in k[t]\langle D_t\rangle$ is a telescoper for F such that $L(F) = D_x(G)$ with $G \in \mathcal{F}[[t]]$, then $L(\operatorname{diag}(f)) = 0$.*

In the following, we prove Furstenberg's diagonal theorem.

Theorem 4.18. *(See [29].) Let $f \in k[[t,x]] \cap k(t,x)$. Then the diagonal of f is a power series algebraic over $k(t)$.*

Proof. Let $F = f(x,t/x)/x$. Since f is a rational function in $k(t,x)$, so is F. Let $L \in k(t)\langle D_t\rangle$ be the minimal telescoper for F. Since multiplying by an element of $k[t]$ commutes with the derivation D_x, we can always assume that the coefficients of L are polynomials in $k[t]$. By Theorem 4.16, all of the solutions of $L(y(t)) = 0$ are algebraic over $k(t)$. So the diagonal of f is algebraic over $k(t)$ since $L(\operatorname{diag}(f)) = 0$ by Lemma 4.17. □

The following example is borrowed from the recent paper by Ekhad and Zeilberger [25], from which one can see how Zeilberger's method of creative telescoping plays a role in solving concrete problems in combinatorics.

Example 4.19. Let $s(n)$ be the number of binary words of length n for which the number of occurrences of 00 is the same as that of 01 as subwords. Stanley [47] asked for a proof of the following formula

$$S(t) \triangleq \sum_{n=0}^{\infty} s(n)t^n = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}}\right). \tag{9}$$

We first show that the generating function $S(t)$ is an algebraic function over $k(t)$. The key ingredient is the Goulden–Jackson cluster method [31]. Noonan and Zeilberger [40] gave an elegant survey of this method together with an efficient implementation. Let $\mathcal{W}$ be the set of all binary words and let $\tau_{00}(w)$, $\tau_{01}(w)$ be the numbers of occurrences of 00 and 01 in $w \in \mathcal{W}$, respectively. Ekhad and Zeilberger [25] define the generating function

$$f(t,y,z) = \sum_{w \in \mathcal{W}} t^{\operatorname{length}(w)} y^{\tau_{00}(w)} z^{\tau_{01}(w)}.$$

Loading the package DAVID_IAN created by Noonan and Zeilberger to Maple, typing GJstDetail([0, 1], {[0, 0], [0, 1]}, t, s), and replacing $s[0,0]$, $s[0,1]$ by y, z, respectively, we get an explicit form of $f(t,y,z)$,

$$f(t,y,z) = \frac{(1-y)t+1}{(y-z)t^2 - (1+y)t + 1},$$

which is a rational function of three variables. By definition, the desired generating function $S(t)$ is the coefficient of x^{-1} in $F(t,x) := x^{-1} f(t,x,x^{-1})$. Since $\tau_{00}(w)$ and $\tau_{01}(w)$ are bounded by $\mathrm{length}(w)$, the function $F(t,x)$ is an element in the ring $k((x))[[t]]$. Therefore, the coefficient $[x^{-1}](F)$ is annihilated by any telescoper for F in $k[t]\langle D_t\rangle$. By Theorem 4.16, the function $S(t)$ must be an algebraic function over $k(t)$. By typing DETools[Zeilberger](F, t, x, Dt) in Maple, we get the minimal telescoper L for F, which is

$$L = \left(-1+5t-13t^2-30t^4+23t^3+40t^5-40t^6+16t^7\right)Dt^2 + \left(80t^6-168t^5+152t^4-88t^3+24t^2-2t+2\right)Dt + 48t^5-72t^4+48t^3-12t^2-6t.$$

To show Stanley's formula (9), it suffices to verify that $S(t)$ satisfies the equation $L(y(t)) = 0$, and check the two initial conditions: $y(0) = 1$ and $D_t(y)(0) = 2$. Moreover, we could also rediscover Stanley's formula by solving the differential equation. Thanks to Zeilberger's method, many classical combinatorial identities now can be proved and rediscovered automatically all by computer.

Except the case when $\partial_t = D_t$ and $\partial_x = D_x$ as above, we will show that telescopers for non-integrable or non-summable rational functions in $k(t,x)$ have at least one nonzero rational solution in $k(t)$. Of these 8 cases, 6 follow easily from an examination of some of the proofs above. These cases are considered in Theorem 4.20. The remaining two cases require a slightly more detailed proof and are considered in Theorem 4.21.

Theorem 4.20. *Let $L \in k(t)\langle\partial_t\rangle$ and $f \in k(t,x)$ satisfy one of the following conditions:*

1. $\partial_t = D_t$ *and* $f \notin \Delta_x(k(t,x))$;
2. $\partial_t = D_t$ *and* $f \notin \Delta_{q,x}(k(t,x))$;
3. $\partial_t = S_t$ *and* $f \notin D_x(k(t,x))$;
4. $\partial_t = S_t$ *and* $f \notin \Delta_{q,x}(k(t,x))$;
5. $\partial_t = Q_t$ *and* $f \notin D_x(k(t,x))$;
6. $\partial_t = Q_t$ *and* $f \notin \Delta_x(k(t,x))$.

Then $L(t,\partial_t)$ is a telescoper for some $f \in k(t,x)$ if and only if $L(y(t)) = 0$ has a nonzero rational solution in $k(t)$.

Proof. Suppose that $L(y(t)) = 0$ has a nonzero rational solution $r(t)$ in $k(t)$. Then L is a telescoper for $f = r(t)/x$ and f satisfies the assumption above. For the opposite implication, Theorems 4.9, 4.13, 4.6, 4.14, 4.7 and 4.12 imply that the residual form of f is of the form a/b such that $b = b_1(t)b_2(x)$ with $b_1 \in k[t]$ and $b_2 \in k[x]$. Then

$$\frac{a}{b} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x-\beta_i)^j},$$

where $\alpha_{i,j} \in k(t)$ and $\beta_i \in k$ are in distinct (q-)orbits. If L is a telescoper for f, then L is also a telescoper for a/b. Since all the β_i are free of t, we have

$$L(a/b) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{L(\alpha i, j)}{(x-\beta_i)^j} = \partial_x(g), \quad \text{where } \partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}.$$

By Propositions 2.2, 2.5, and 2.10, we have $L(\alpha_{i,j}) = 0$. Since a/b is not zero, at least one of the $\alpha_{i,j}$ is nonzero. Thus $L(y(t)) = 0$ has at least one nonzero rational solution in $k(t)$. □

Theorem 4.21. *Let $L \in k(t)\langle\partial_t\rangle$ and $f \in k(t,x)$ satisfy one of the following conditions*: (1) $\partial_t = S_t$ *and* $f \notin \Delta_x(k(t,x))$; (2) $\partial_t = Q_t$ *and* $f \notin \Delta_{q,x}(k(t,x))$. *Then $L(t,\partial_t)$ is a telescoper for some $f \in k(t,x)$ if and only if $L(y(t)) = 0$ has a nonzero rational solution in $k(t)$.*

Proof. Suppose that $L(y(t)) = 0$ has a nonzero rational solution $r(t)$ in $k(t)$. Then L is a telescoper for $f = r(t)/x$ and f satisfies the assumption above. For the opposite implication, we only prove the assertion for the first case, that is, when L and f satisfies the condition (1). The remaining assertion follows in a similar manner. Theorem 4.11 implies that the residual form a/b of f can be decomposed into

$$\frac{a}{b} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(x-\beta_i)^j},$$

where $\alpha_{i,j} \in k(t)$ and $\beta_i = \frac{\lambda_i}{\mu_i}t + c_i$ with $c_i \in k$, $\lambda_i \in \mathbb{Z}$ and $\mu_i \in \mathbb{N}$ such that $\gcd(\lambda_i,\mu_i) = 1$ and the β_i are in distinct $\mathbb{Z}$-orbits. If $L \in k(t)\langle D_t\rangle$ is a telescoper for f, then L is a telescoper for a/b. Moreover, L is a telescoper for each fraction $f_{i,j} = \alpha_{i,j}/(x-\beta_i)^j$. We claim that the operator $L_{i,j} := \alpha_{i,j}(t)S_t^{\mu_i} - \alpha_{i,j}(t+\mu_i) \in k(t)\langle D_t\rangle$ is the minimal telescoper for $f_{i,j}$ with respect to Δ_x. In fact, $L_{i,j}$ is a telescoper for $f_{i,j}$ as shown in the proof of Theorem 4.11. It remains to show the minimality. Assume that there exists a telescoper $\tilde{L}_{i,j}$ of order less than μ_i for $f_{i,j}$. Write $\tilde{L}_{i,j} = \sum_{\ell=0}^{\mu_i-1} e_\ell S_t^\ell$. Then

$$\tilde{L}_{i,j}(f_{i,j}) = \sum_{\ell=0}^{\mu_i-1} \frac{e_\ell \alpha_{i,j}(t+\ell)}{(x-(\frac{\lambda_i}{\mu_i}t + \frac{\lambda_i}{\mu_i}\ell + c_i))^j}.$$

Since $\gcd(\lambda_i,\mu_i) = 1$ and $\ell \in \{0,\ldots,\mu_i-1\}$, the values $\frac{\lambda_i}{\mu_i}t + \frac{\lambda_i}{\mu_i}\ell + c_i$ are in distinct $\mathbb{Z}$-orbits. If $\tilde{L}_{i,j}(f_{i,j})$ is rational summable, then all the residues $e_\ell\alpha_{i,j}(t+\ell)$ are zero by Proposition 2.5. Since $\alpha_{i,j} \neq 0$, we have $\tilde{L}_{i,j}$ is a zero operator. The claim holds. Since L is a telescoper for $f_{i,j}$, $L_{i,j}$ divides L on the right. Note that the rational function $\alpha_{i,j} \in k(t)$ is a nonzero solution of $L_{i,j}(y(t)) = 0$. Thus, L has at least one nonzero rational solution in $k(t)$. □

Appendix A

In this appendix, we present proofs of Propositions 3.2 and 3.3. Let $K \subset E$ be difference fields of characteristic zero with automorphism σ and assume that the constants E^σ of E are in K. Furthermore assume that E is algebraically closed.

Lemma A.1. *Let $u \in E$ be algebraic over K and assume that u satisfies a homogeneous linear difference equation over K. Then there exists a field $F \subset E$ with $\sigma(F) = F$, $K \subset F$, $[F:K] < \infty$, and $u \in F$.*

Proof. Let u satisfy

$$\sigma^n(u) + b_{n-1}\sigma^{n-1}(u) + \cdots + b_0 u = 0 \tag{10}$$

with $b_i \in K$, $b_0 \neq 0$ and let $F = K(u,\sigma(u),\ldots,\sigma^{n-1}(u))$. We have that $[F:K] < \infty$ since for any i, $\sigma^i(u)$ is algebraic over K. To see that $\sigma(F) \subset F$ it is enough to show that $\sigma^i(u) \in F$ for all i. This is certainly true for $i = 0,\ldots,n$. If $i > n$, apply σ^{i-n} to Eq. (10) and proceed by induction to conclude $\sigma^i(u) \in F$. If $i < 0$ apply σ^i and proceed by induction to conclude $\sigma^i(u) \in F$. □

Lemma A.2. *Let $K = k(t)$, where k is algebraically closed. Let (E,σ) be a difference field such that $K \subset E$, $\sigma(t) = t+1$ and $[E:K] < \infty$. The $E = K$.*

Proof. Let $n = [E:K]$ and g be the genus of E. The Riemann–Hurwitz formula (see [18, p. 106] or [28, p. 125]) yields

$$2g - 2 = -2n + \sum_P \bigl(e(P) - 1\bigr), \tag{11}$$

where the sum is over all places P of E and $e(P)$ is the ramification index of P with respect to K. There are only a finite number of places Q of K over which places of E ramify and the automorphism σ leaves the set of such places invariant. On the other hand, the only finite set of places of K that is left invariant by σ is the place at infinity. Therefore, if P is a place of E with $e(P) > 1$, then P lies above the place at infinity. Note that for any place Q of K, Theorem 1 of [18, p. 52] implies (under our assumptions) that

$$\sum_{P \text{ lies above } Q} e(P) = n. \tag{12}$$

Therefore we have

$$\begin{aligned} 2g - 2 &= -2n + \sum_{P \text{ lies above } \infty} \bigl(e(P) - 1\bigr) \\ &= -2n + n - t \\ &= -n - t, \end{aligned}$$

where t is the number of places above infinity. Since n and t are both positive integers and g is nonnegative, we must have $g = 0$ and $n = t = 1$. In particular, since $n = 1$, we have $E = K$. □

Proof of Proposition 3.2. Suppose that $\alpha(t)$ satisfies the linear recurrence relation

$$S_t^n(\alpha) + a_{n-1} S_t^{n-1}(\alpha) + \cdots + a_0 \alpha = 0,$$

where $a_i \in k(t)$. By Lemma A.1, the field $E = k(t)(\alpha, S_t(\alpha), \ldots, S_t^{n-1}(\alpha)) \subset \overline{k(t)}$ is a difference field extension of $k(t)$. Since $[E : k(t)] < \infty$, $E = k(t)$ by Proposition A.2. Thus $\alpha \in k(t)$. □

Remark A.3. Proposition 3.2 has been shown in [12, Theorem 1], [51, Proposition 4.4] and [11, Theorem 5.2]. The proof in [11, Theorem 5.2] is based on analytic properties of algebraic functions.

In this proposition, we assume that $\alpha(t)$ satisfies a polynomial equation over $k(t)$ and *lies in a field*. This latter condition cannot be weakened without weakening the conclusion. For example, the sequence $y = (-1)^n$ satisfies $y^2 - 1 = 0$ but $k(t)[y]$ is a ring with zero divisors. The above references give a complete characterization of sequences satisfying both linear recurrences and polynomial equations.

The following result is a q-analogue of Lemma A.2.

Lemma A.4. *Let $K = k(t)$, where k is algebraically closed. Let (E, σ) be a difference field such that $K \subset E$, $\sigma(t) = qt$ with $q \in k \setminus \{0\}$ and not a root of unity, and $[E : K] < \infty$. Then $E = k(t^{1/n})$ for some positive integer n.*

Proof. Let $[E : K] = n$ and g be the genus of E. We again consider the set of places of K over which places of E ramify. This set is left invariant by σ and so must be a subset of the set containing the place at 0 and the place at ∞. Therefore, ramification can occur only at 0 and ∞. Eqs. (11) and (12) imply

$$\begin{aligned} 2g-2 &= -2n + \sum_{P \text{ lies above } 0} \bigl(e(P)-1\bigr) + \sum_{P \text{ lies above } \infty} \bigl(e(P)-1\bigr) \\ &= -2n + 2n - t_0 - t_\infty \\ &= -t_0 - t_\infty \end{aligned}$$

where t_0, t_∞ are the numbers of places above 0 and ∞. Since t_0 and t_∞ are positive and g is nonnegative, we must have that $g=0$ and $t_0 = t_\infty = 1$. Therefore, E has one place P_0 over 0 with $e_{P_0} = n$ and one place P_∞ over ∞ with $e_{P_\infty} = n$. Writing divisors multiplicatively, Riemann's theorem [18, p. 22] implies that

$$l\bigl(P_0 P_\infty^{-1}\bigr) \geqslant d\bigl(P_0 P_\infty^{-1}\bigr) - g + 1 = 0 - 0 + 1 = 1$$

where $l(P_0 P_\infty^{-1})$ is the dimension of the space of elements of E which are $\equiv 0 \bmod P_0 P_\infty^{-1}$. Note that since the degree $P_0 P_\infty^{-1}$ is 0, this latter condition implies that any such element has $P_0 P_\infty^{-1}$ as its divisor. Therefore, there exists an element $y \in E$ whose divisor is $P_0 P_\infty^{-1}$. Note that the element t has divisor $P_0^n P_\infty^{-n}$ and therefore $y^n t^{-1}$ must be in k. Therefore $y = ct^{1/n}$ for some $c \in k$. Finally, Theorem 4 of [18, p. 18] states that $[E : k(y)]$ equals the degree of the divisor of zeros of y, that is, $[E : k(y)] = 1$. Therefore $E = k(y) = k(t^{1/n})$. $\square$

Proof of Proposition 3.3. Suppose that $\alpha(t)$ satisfies the linear q-recurrence relation

$$Q_t^n(\alpha) + a_{n-1} Q_t^{n-1}(\alpha) + \cdots + a_0 \alpha = 0,$$

where $a_i \in k(t)$. By Lemma A.1, the field $E = k(t)(\alpha, Q_t(\alpha), \ldots, Q_t^{n-1}(\alpha)) \subset \overline{k(t)}$ is a difference field extension of $k(t)$. Since $[E : k(t)] < \infty$, $E = k(t^{1/n})$ by Lemma A.4. Thus $\alpha \in k(t^{1/n})$. $\square$

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