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# PROJECTIVE ISOMONODROMY AND GALOIS GROUPS

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Abstract. In this article we introduce the notion of projective isomonodromy, which is a special type of monodromy evolving deformation of linear differential equations, based on the example of the Darboux-Halphen equation. We give an algebraic condition for a paramaterized linear differential equation to be projectively isomonodromic, in terms of the derived group of its parameterized Picard-Vessiot group.

## 1. Introduction

Classical, monodromy preserving deformations of Fuchsian systems have been investigated by many authors who described them in terms of the Schlesinger equation and its links to Painlevé equations. In [8], Cassidy and Singer have developed a new Galois theory for parameterized linear differential equations and defined their parameterized Picard-Vessiot group, PPV-group for short. This is a linear differential algebraic group in the sense of Cassidy [6]. As is well known, the differential Galois group of a system with regular singularities is, as a linear algebraic group, Zariski topologically generated by the monodromy matrices with respect to a fundamental solution. Cassidy and Singer have shown that a parameterized family of such systems is isomonodromic if and only if its PPV-group is conjugate to a (constant) linear algebraic group.

Analogous to the Schlesinger and Painlevé equations' relation to isomonodromic deformations of Fuchsian systems, the Darboux-Halphen V equation accounts for a special type of monodromy evolving deformation of Fuchsian systems, as was shown by Chakravarty and Ablowitz in [9]. In this article we first describe the Darboux-Halphen system, then define the general notion of projective isomonodromy illustrated by this example. We characterize projective isomonodromy in different ways, by a condition on the residue matrices for families of Fuchsian systems, and by the condition that the derived group $(G, G)$ of the PPV-group $G$ be conjugate to a constant linear algebraic group when the given equation is absolutely irreducible.

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## 2. Classical isomonodromy

In the classical study of isomonodromic deformations, only parameterized Fuchsian systems are considered. Furthermore, these systems are assumed to be parameterized in a very special way, that is, the systems are written as

$$\frac{dY}{dx} = \sum_{i=1}^{m} \frac{A_i(a)}{x-a_i}, \quad \sum_{i=1}^{m} A_i(a) = 0 \tag{2.1}$$

where the $n \times n$ matrices $A_i(a)$ depend holomorphically on the multi-parameter $a = (a_1, \ldots, a_n)$ in some open polydisk $D(a^0)$ and the condition on the residue matrices guarantees, for simplicity, that $\infty$ is not singular. The polydisk $D(a^0) = D_1 \times \ldots \times D_m$ has center at the initial location $a^0 = (a_1^0, \ldots, a_m^0) \in \mathbb{C}^m$ of the poles, with $D_i \subset \mathbb{C}$ a disk with center $a_i^0$ and $D_i \cap D_j \neq \emptyset$ for all $i \neq j$. Let $x_0 \in \mathcal{D} = \mathbb{P}^1(\mathbb{C}) \setminus \bigcup_i D_i$.

For fixed $a \in D(a^0)$ and local fundamental solution $Y_a$ of (2.1) at $a$, analytic continuation along a loop $\gamma$ in $\mathcal{D}_a = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \ldots, a_m\}$ yields a solution $Y_a^\gamma$. The *monodromy representation* with respect to $Y_a$ is

$$\chi_a : \pi_1(\mathcal{D}_a; x_0) \to \mathrm{GL}_n(\mathbb{C}) \tag{2.2}$$

defined by

$$Y_a^\gamma = Y_a.\chi_a(\gamma),$$

for all $[\gamma] \in \pi_1(\mathcal{D}_a; x_0)$.

**Definition 1.** Equation (2.1) is *isomonodromic*, or an *isomonodromic deformation*, if for all $a \in D(a^0)$ there are matrices $C(a) \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\chi_a = C(a)\ \chi_{a^0}\ C(a)^{-1}.$$

Bolibrukh ([3], [4]) has characterized isomonodromic deformations as follows.

**Theorem 2** (Bolibrukh)**.** Equation (2.1) is isomonodromic if and only if the following equivalent conditions hold

(1) There is a differential 1-form $\omega$ on $\mathbb{P}^1(\mathbb{C}) \times D(a^0) \setminus \bigcup_{i=1}^m \{(x,a) \mid x - a_i = 0\}$ such that
  - for each fixed $a \in D(a^0)$,
  $$\omega = \sum_{i=1}^{m} \frac{A_i(a)}{x-a_i} dx$$
  - $d\omega = \omega \wedge \omega$.

(2) For each $a \in D(a^0)$ there is a fundamental solution $Y_a$ of (2.1) such that $Y_a(x)$ is analytic in $x$ and $a$, and the corresponding monodromy representation $\chi_a$ does not depend on $a$, that is, $\chi_a = \chi_{a^0}$.

A special type of isomonodromic deformation is given by the Schlesinger differential form

$$\omega_s = \sum_{i=1}^{m} \frac{A_i(a)}{x-a_i} d(x-a_i) \tag{2.3}$$

whose integrability condition is known as the *Schlesinger equation*

$$dA_i(a) = -\sum_{j=1, j\neq i}^{m} \frac{[A_i(a), A_j(a)]}{a_i - a_j} d(a_i - a_j), \quad i = 1, \dots m. \tag{2.4}$$

Bolibrukh gave examples [3] of isomonodromic deformations that are not of the Schlesinger type and he described the general differential forms that occur in Theorem (2).

In the special case of order two Fuchsian systems with four singularities one can, generically, reduce each system to an order two linear scalar differential equation satisfied by the first component of the dependent variable $Y$, namely a Fuchsian scalar equation with an additional apparent singularity $\lambda$. It is well known that the Schlesinger isomonodromy condition then translates into a non-linear equation of Painlevé VI type satisfied by $\lambda$. For basic results about Painlevé equations and isomonodromic deformations, we refer to [12] and [1] .

## 3. An example of a monodromy evolving deformation

In [9], Chakravarty and Ablowitz describe the Darboux-Halphen system

$$\begin{cases} \omega_1' &= & \omega_2\omega_3 & - & \omega_1(\omega_2+\omega_3) & + & \phi^2 \\ \omega_2' &= & \omega_3\omega_1 & - & \omega_2(\omega_3+\omega_1) & + & \theta^2 \\ \omega_3' &= & \omega_1\omega_2 & - & \omega_3(\omega_1\omega_2) & - & \theta\phi \\ \phi' &= & \omega_1(\theta-\phi) & - & \omega_3(\theta+\phi) & & \\ \theta' &= & -\omega_2(\theta-\phi) & - & \omega_3(\theta+\phi) & & \end{cases} \tag{3.1}$$

as a prototype of a class of non-linear systems arising as the integrability conditions of an associated Lax pair in the same way as the Painlevé and Schlesinger equations do. This system occurs in the Bianchi IX cosmological models and arises from a special reduction of the self-dual Yang-Mills (SDYM) equation (*cf.* [1], [9], [10]). It is also related to the Chazy and Painlevé VI equations (see [1] for a complete study of such equations and reductions of the SDYM equation). We will follow Ohyama [16] who studied this equation in more details, and refer to (3.1) as the Darboux-Halphen V equation or DH-V for short.

Originally (*cf.* [9]) the DH-V system with the special condition $\phi = \omega_3 = 0$ arose from a geometrical problem studied by Darboux, who in 1878 obtained it as the integrability condition for the existence in Euclidean space of a one-parameter family of surfaces of second degree orthogonal to two arbitrary given independent families of parallel surfaces. Halphen solved this system in 1881.

Ohyama ([16], [17]) shows how DH-V is, in the generic case, equivalent to Halphen's second equation H-II

$$x_i' = Q(x_i), \quad i = 1, 2, 3 \tag{3.2}$$

where

$$Q(x) = x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2 \tag{3.3}$$

with constants $a, b, c$ such that $a+b = c+b = -1/4$ (all derivatives are with respect to the complex variable $t$).

As pointed out in [16], these equations do not satisfy the Painlevé property (for their movable singularities) and may therefore not be expected to be monodromy-preserving conditions. Nevertheless Chakravarty and Ablowitz [9] showed how these non-linear equations actually express a special type of monodromy evolving deformation, in the same way as the Schlesinger and Painlevé VI equations rule the isomonodromic deformations of the Schlesinger type.

Using the connection relating the self-dual Yang-Mills equation and the conformally self-dual Bianchi equations, Chakravarty and Ablowitz [9], followed by Ohyama [16], showed that DH-V, and hence H-II, actually is the compatibility condition of a Lax pair

$$\frac{\partial Y}{\partial x} = \left(\frac{\mu}{P}I + \sum_{i=1}^{3}\frac{\lambda_i S}{x - x_i}\right)Y \tag{3.4}$$

$$\frac{\partial Y}{\partial t} = \left(\nu I + \sum_{i=1}^{3}\lambda_i x_i S\right)Y - Q(x)\frac{\partial Y}{\partial x} \tag{3.5}$$

of $2 \times 2$ matrix equations, where $x_1, x_2, x_3$ depend on $t$ and $P(x) = (x - x_1)(x - x_2)(x - x_3)$, and $S$ is a traceless constant matrix (the diagonal entries are equal to zero), and $\mu$ and the $\lambda_i$'s are constants with $\mu \neq 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and $\nu(x, t)$ satisfies the auxiliary equation

$$\frac{\partial \nu}{\partial x} = -\frac{x + x_1 + x_2 + x_3}{P}\mu. \tag{3.6}$$

Under these asumptions, (3.4) is for fixed $t$ a Fuchsian system with three singular points $x_1, x_2, x_3$, and Equation (3.6) implies that $\nu$ is not a rational function of $x$. Therefore the Lax pair (3.4), (3.5) does not describe an isomonodromic deformation, since otherwise the coefficients of (3.5) would be rational (*cf.* [19], Remark A.5.2.5).

Let us fix $t_0 \in \mathbb{C}$, and small open disjoint disks $D_i$ with center at $x_i(t_0)$, $i = 1, 2, 3$. Let $U(t_0)$ be a neighborhood of $t_0$ in $\mathbb{C}$ such that $x_i(t) \in D_i$ for each $i$ and all $t \in U(t_0)$, and let $x_0 \in \mathbb{C}$ be a fixed base-point, $x_0 \notin \bigcup_i D_i$.

Let $Y(t, x)$, for $t \in U(t_0)$, denote a fundamental solution, in a neighborhood of $x_0$, of the Lax pair ((3.4), (3.5)). It is therefore analytic in both $t$ and $x$. For fixed $t \in U(t_0)$, we can write an analytic continuation of the fundamental solution $Y(t, x)$ to a punctured neighborhood of $x_i$ as

$$Y(t, x) = Y_i(t, x - x_i(t)).(x - x_i(t))^{L_i(t)} \tag{3.7}$$

where $Y_i(t, x - x_i(t))$ is single-valued, and the matrix $L_i(t)$ does not depend on $x$. Note that $Y_i(t, x - x_i(t))$ is analytic in $t$ and $x$ and $L_i(t)$ is analytic in $t$. Indeed, for fixed $t \in U(t_0)$, analytic continuation of $Y$ along an elementary loop around $x_i(t)$ yields a fundamental solution $\tilde{Y}(t, x)$ of (3.4) which is again analytic in both $t$ and $x$, by the theorem about analytic dependence on initial conditions (*cf.* [5]). The monodromy matrix $M_i(t)$ is therefore analytic in $t$, as well as $L_i = (1/2\pi i)\log M_i(t)$, and hence $Y_i(t, x - x_i(t)) = Y(t, x)\ (x - x_i(t))^{-L_i(t)}$ is analytic in $t$ and $x$ in $U(t_0) \times D_i \setminus \{(t, x) \mid x - x_i(t) = 0\}$.

**Proposition 3.** With notation as above, let $M_i(t)$ for any fixed $t \in U(t_0)$ denote the monodromy matrix of (3.4) with respect to $Y$, defined by analytic continuation along an elementary loop around $x_i(t)$. Then

$$M_i(t) = c_i(t)G_i \tag{3.8}$$

where $G_i$ is a constant matrix and $c_i(t) = e^{-2\pi\mu\sqrt{-1}\int_{t_0}^t \alpha_i(t)dt}$ and the $\alpha_i$ are the residues of

$$\frac{x + x_1 + x_2 + x_3}{P} = \sum_{i=1}^{3} \frac{\alpha_i}{x - x_i}. \tag{3.9}$$

*Proof.* Let us show that

$$\frac{dL_i}{dt} = -\alpha_i \mu I \tag{3.10}$$

where $\alpha_i$ is the $x_i$-residue of $(x + x_1 + x_2 + x_3)/P$, that is,

$$\alpha_i = \frac{x_i + \sigma}{\prod_{j\neq i}(x_i - x_j)}$$

with $\sigma = x_1 + x_2 + x_3$. Note that $Y_i$ is a function of the local coordinate $x - x_i$, hence

$$\partial Y_i/\partial x_i = -\partial Y_i/\partial x.$$

We have

$$\frac{\partial Y}{\partial x_i} = -\frac{\partial Y_i}{\partial x}(x - x_i)^{L_i} + Y_i(x - x_i)^{L_i}\left(\frac{\partial L_i}{\partial x_i}\log(x - x_i) - \frac{L_i}{x - x_i}\right)$$

$$= \left(-\frac{\partial Y_i}{\partial x}(x - x_i)^{L_i} - Y_i(x - x_i)^{L_i}\frac{L_i}{x - x_i}\right) + Y_i(x - x_i)^{L_i}\left(\frac{\partial L_i}{\partial x_i}\log(x - x_i)\right)$$

$$= -\frac{\partial Y}{\partial x} + Y_i(x - x_i)^{L_i}\left(\frac{\partial L_i}{\partial x_i}\log(x - x_i)\right),$$

and

$$\frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial x_i}\frac{\partial x_i}{\partial t} = Q(x_i)\frac{\partial Y}{\partial x_i} = -\frac{\partial Y}{\partial x}Q(x_i) + Y\left(\frac{\partial L_i}{\partial x_i}\log(x - x_i)\right)Q(x_i).$$

For any fixed $t$, this is also equal to (see Equation (3.5) of the Lax pair)

$$\frac{\partial Y}{\partial t} = -\frac{\partial Y}{\partial x}Q(x) + \left(\nu I + \sum_{i=1}^{3} c_i x_i S\right)Y$$

and as

$$\frac{dL_i}{dt} = Q(x_i)\frac{dL_i}{dx_i},$$

(we abusively use the same notation for $L_i$ as a function of $x_i$ and $L_i$ as a function of $t$ via $x_i(t)$), comparing the two expressions we get

$$-\frac{\partial Y}{\partial x}Q(x_i) + Y\left(\frac{dL_i}{dt}\log(x - x_i)\right) = -\frac{\partial Y}{\partial x}Q(x) + \left(\nu I + \sum_{i=1}^{3} c_i x_i S\right)Y.$$

From Equation (3.6) we have that

$$\nu = \mu \log \prod_{i=1}^{3}(x - x_i)^{-\alpha_i} + \phi(t)$$

for some function $\phi(t)$, and hence as $x$ tends to $x_i$ (simplifying and then comparing the leading terms on each side) we get that

$$\frac{dL_i}{dt}\log(x - x_i) \sim -\alpha_i \mu \log(x - x_i),$$

that is,

$$\frac{dL_i}{dt} = -\alpha_i \mu I.$$

The monodromy matrix of (3.4) with respect to $x_0$ and $Y$ around $x_i$ is $M_i = e^{2\pi i L_i}$, which in view of (3.10) is of the form

$$M_i(t) = c_i(t) G_i$$

where $G_i$ is the initial monodromy matrix around $x_i(t_0)$, and $c_i(t) = e^{-2\pi\mu\sqrt{-1}\int_{t_0}^{t} \alpha_i(t)dt}$.

□

This is an example of what we will call projectively isomonodromic deformations and study from an algebraic point of view.

## 4. Projective isomonodromy

Let $\mathcal{D}$ be an open connected subset of $\mathbb{P}^1(\mathbb{C})$, $\mathcal{P}$ be an open connected subset of $\mathbb{C}^r$, and $x_0 \in \mathcal{D}$. Assume that $\pi_1(\mathcal{D}, x_0)$ is finitely generated by $\gamma_1, \ldots, \gamma_m$. Let $A(x,t) \in \mathrm{GL}_n(\mathcal{O})$, the ring of $n \times n$ matrices whose entries are functions analytic on $\mathcal{O} = \mathcal{D} \times \mathcal{P}$. We will consider the behavior of solutions of the differential equation

$$\frac{dY}{dx} = A(x,t)Y. \tag{4.1}$$

In the following we let $\mathrm{Scal}_n$ be the group of nonzero $n \times n$ scalar matrices.

**Definition 4.** Equation (4.1) is *projectively isomonodromic* if there exist $m$ analytic functions $c_i : \mathcal{P} \to \mathrm{Scal}_n(\mathbb{C})$ and fixed matrices $G_1, \ldots, G_m \in \mathrm{GL}_n(\mathbb{C})$ such that for each $t \in \mathcal{P}$ there is a local solution $Y_t(x)$ of (4.1) at $x_0$ such that analytic continuation of $Y_t(x)$ along $\gamma_i$ yields $Y_t(x) \cdot G_i \cdot c_i(t)$, for each $i$.

Let $\bar{Y}(x,t)$ be any solution of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where $\mathcal{D}_0$ is a neighborhood of $x_0$ in $\mathcal{D}$ and let $G_i(t)$ denote the monodromy matrix corresponding to analytic continuation of this solution around $\gamma_i$. Note that $G_i(t)$ depends analytically on $t$. If (4.1) is projectively isomonodromic then there exists a function $C(t) : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ such that

$$G_i(t) = C(t)^{-1} G_i c_i(t) C(t).$$

Since there may be many ways of selecting $C(t)$, this function need not depend analytically on $t$. However, we will show that one can find a function $C(t)$ satisfying the above *and analytic in* $t$. This fact can be deduced easily from the following result of Andrey Bolibruch whose proof is contained in the proof of Proposition 1 of [3].

**Proposition 5.** For each $i = 1, \ldots, m$, let $H_i(t) : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ be analytic on $\mathcal{P}$ and let $G_i \in \mathrm{GL}_n(\mathbb{C})$. Assume that there is a function $C(t) : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ such that

$$H_i(t) = C(t)^{-1} G_i C(t)$$

for all $t \in \mathcal{P}$ and $i = 1, \ldots, m$. Then there exists an analytic function $C(t)$ with the same property.

We can now prove the following

**Proposition 6.** If (4.1) is projectively isomonodromic, then there exists a solution $Y(x,t)$ of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where $\mathcal{D}_0$ is a neighborhood of $x_0$ in $\mathcal{D}$ such that for all $t \in \mathcal{P}$ the monodromy matrix of $Y(x,t)$ along $\gamma$ $\underline{\text{is}}$ $G_i \cdot c_i(t)$.

*Proof.* Let $\bar{Y}(x,t)$ be any solution of (4.1) analytic in $\mathcal{D}_0 \times \mathcal{P}$, where $\mathcal{D}_0$ is a neighborhood of $x_0$ in $\mathcal{D}$ and let $G_i(t)$ denote the monodromy matrix corresponding to analytic continuation of this solution around $\gamma_i$. Since (4.1) is projectively isomonodromic, there is a function $C(t) : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ such that $G_i(t) = C(t)^{-1} G_i c_i(t) C(t)$. Applying Proposition 5 to $H_i(t) = G_i(t) c_i^{-1}(t)$ and $G_i$, we may assume that $C(t)$ is analytic and thus $Y(x,t) = \bar{Y}(x,t) \cdot C(t)$ satisfies the conclusion of this Proposition. □

## 5. Isomonodromy versus projective isomonodromy

We now turn to the relation between Fuchsian isomonodromic equations and Fuchsian projectively isomonodromic equations. Consider the equation

$$\frac{dY}{dx} = \sum_{i=1}^{m} \frac{A_i(t)}{x - x_i(t)} Y \tag{5.1}$$

together with

(1) $\mathcal{P}$, a simply connected open subset of $\mathbb{C}^r$ and
(2) $\mathcal{D}$, an open subset in $\mathbb{P}^1(\mathbb{C})$ and $x_0 \in \mathcal{D}$

such that

(1) the functions $A_i(t) : \mathcal{P} \to \mathrm{gl}_n(\mathbb{C})$ and the $x_i(t) : \mathcal{P} \to \mathbb{C}$ are analytic functions,
(2) $\mathbb{P}^1(\mathbb{C}) \backslash \mathcal{D}$ is the union of $m$ disjoint closed disks and
(3) for $t \in \mathcal{P}$ we have $x_i(t) \in D_i$.

Let $x_0 \in \mathcal{D}$ and $\gamma_i$, $i = 1, \ldots, m$, be the obvious loops generating $\pi_1(\mathcal{D}, x_0)$. We then have that Equation (5.1) is analytic in $\mathcal{D} \times \mathcal{P}$ and we can speak of monodromy matrices $G_i(t)$ corresponding to analytic continuation of a fundamental solution matrix along $\gamma_i$. We can now state

**Proposition 7.** Let $\mathcal{D}$ and $\mathcal{P}$ be as above. Equation (5.1) is projectively isomonodromic if and only if for each $i = 1, \ldots, m$, there exist functions $b_i : \mathcal{P} \to \mathbb{C}$ and $B_i : \mathcal{P} \to \mathrm{gl}_n(\mathbb{C})$, analytic on $\mathcal{P}$ such that

(1) $A_i(t) = B_i(t) + b_i(t) I_n$ for $i = 1, \ldots, m$ and
(2)

$$\frac{dY}{dx} = \left( \sum_{i=1}^{m} \frac{B_i(t)}{x - x_i(t)} \right) Y \tag{5.2}$$

is isomonodromic.

*Proof.* Assume that Equation (5.1) is projectively isomonodromic and let $Y(x,t), G_i$ and $c_i(t)$ be as in the conclusion of Proposition 6. Since $\mathcal{P}$ is simply connected and the $c_i(t)$ are nonzero, there exist analytic $b_i(t) : \mathcal{P} \to \mathbb{C}$ such that $e^{2\pi\sqrt{-1} b_i(t)} = c_i(t)$ for each $i$. Let

$$Z(x,t) = Y(x,t) \cdot \prod_{i=1}^{m} (x - x_i(t))^{-b_i(t)} I_n$$

One sees that the monodromy of $Z$ along $\gamma_i$ is given by $G_i$ and so is independent of $t$. Therefore, letting $B_i(t) = A_i(t) - b_i(t) I_n$, we have that

$$\frac{dY}{dx} = \left(\sum_{i=1}^{s} \frac{B_i(t)}{x - x_i(t)}\right) Y$$

is isomonodromic.

Now assume that $A_i, B_i, b_i$ are as in items 1. and 2. of the proposition and that Equation (5.2) is isomonodromic. If $Y(x,t)$ is a local solution of (5.2) with constant monodromy matrices $G_i$ along $\gamma_i$, then $Z(x,t) = Y(x,t) \cdot \prod_{i=1}^{m}(x - x_i(t))^{b_i(t)} I_n$ will have monodromy $c_i(t)G_i$ along $\gamma_i$, with $c_i(t) = e^{2\pi\sqrt{-1}b_i(t)}$ . Thus Equation (5.1) is projectively isomonodromic. □

Proposition 7 applies to the DH-V example of Chakravarty and Ablowitz, since we can rewrite Equation (3.4) of the Lax pair as

$$\frac{\partial Y}{\partial x} = \left(\sum_{i=1}^{3} \frac{A_i(t)}{(x - x_i)}\right) Y$$

where

$$A_i(t) = B_i(t) + b_i(t) I_n$$

with

$$B_i(t) = \lambda_i S$$
$$b_i(t) = \frac{\mu}{\prod_{i \neq j}(x_i - x_j)}.$$

An easy computation shows that since $x_i' - x_j' = Q(x_i) - Q(x_j) = x_i^2 - x_j^2$ for all $i, j$, we have

$$b_i' = \frac{db_i}{dt} = -\frac{x_i + \sigma}{\prod_{i \neq j}(x_i - x_j)} \mu == -\alpha_i \mu.$$

and we recover the result of Proposition 3, that the monodromy of this equation is evolving 'projectively' and equal to

$$M_i(t) = e^{2\pi\sqrt{-1}b_i(t)} G_i = e^{-2\pi\mu\sqrt{-1}\int_{t_0}^{t} \alpha_i(t)dt} G_i.$$

## 6. Parameterized differential Galois groups

In this section we examine the parameterized differential Galois groups of projectively isomonodromic equations. Parameterized differential Galois groups (*cf.* [8], [15]) generalize the concept of differential Galois groups of the classical Picard-Vessiot theory and we begin this section by briefly describing the underlying theory.

Let

$$\frac{dY}{dx} = A(x) Y \qquad (6.1)$$

be a differential equation where $A(x)$ is an $n \times n$ matrix with entries in $\mathbb{C}(x)$. The usual existence theorems for differential equations imply that if $x = x_0$ is a point in $\mathbb{C}$ such that the entries of $A(x)$ are analytic at $x_0$, then there exists a nonsingular matrix $Z = (z_{i,j})$ of functions analytic in a neighborhood of $x_0$ such that $\frac{dZ}{dx} = A(x)Z$. Note that the field $K = \mathbb{C}(z_{1,1}, \ldots, z_{n,n})$ is closed with respect to taking the derivation $\frac{d}{dx}$ and this is an example of a *Picard-Vessiot extension*[1]. The set of field-theoretic isomorphisms of $K$ that leave $\mathbb{C}(x)$ elementwise-fixed and commute with $\frac{d}{dx}$ forms a group $G$ called the *Picard-Vessiot group* or *differential Galois group* of (6.1). One can show that for any $\sigma \in G$, there exists a matrix $M_\sigma \in \mathrm{GL}_n(\mathbb{C})$ such that $\sigma(Z) = (\sigma(z_{i,j})) = ZM_\sigma$. The map $\sigma \mapsto M_\sigma$ is an isomorphism whose image is furthermore a *linear algebraic group*, that is, a group of invertible matrices whose entries satisfy some fixed set of polynomial equations in $n^2$ variables. There is a well developed Galois theory for these groups that identifies certain subgroups of $G$ with certain subfields of $K$ and associates properties of the equation (6.1) with properties of the groups $G$. The elements of the monodromy group of (6.1) may be identified with elements of this group and, when (6.1) has only regular singular points, it is known that $G$ is the smallest linear algebraic group containing these elements (*cf.* [18], Theorem 5.8). Further facts about this Galois theory can be found in [13] and [18].

Now let

$$\frac{dY}{dx} = A(x,t)Y \tag{6.2}$$

be a parameterized system of linear differential equations where $A(x,t)$ is an $n \times n$ matrix whose entries are rational functions of $x$ with coefficients that are functions of $t = (t_1, \ldots, t_r)$, analytic in some domain in $\mathbb{C}^r$. A differential Galois theory for such equations was developed in [8] and in greater generality in [15]. Let $k_0$ be a suitably large field[2] containing $\mathbb{C}(t_1, \ldots, t_r)$ and the functions of $t$ appearing as coefficients in the entries of $A$ and such that $k_0$ is closed under the derivations $\Pi = \{\partial_1, \ldots, \partial_r\}$ where each $\partial_i$ restricts to $\frac{\partial}{\partial t_i}$ on $\mathbb{C}(t_1, \ldots, t_r)$ and the intersection of the kernels of the $\partial_i$ is $\mathbb{C}$. As before, existence theorems for solutions of differential equations guarantee the existence of a nonsingular matrix $Z(x,t) = (z_{i,j}(x,t))$ of functions, analytic in some suitable domain in $\mathbb{C} \times \mathbb{C}^r$, such that $\frac{dZ}{dx} = AZ$. We will let $k = k_0(x)$ be the differential field with derivations $\Delta = \{\partial_x, \partial_1, \ldots, \partial_r\}$ where $\partial_x(x) = 1$, $\partial_x(z) = 0$ for all $z \in k_0$ and the $\partial_i$ extend the previous $\partial_i$ with $\partial_i(x) = 0$. Finally we will denote by $K$ the smallest field containing $k$ and the $z_{i,j}$ that is closed under the derivations of $\Delta$. This field is called the *parameterized Picard-Vessiot field* or PPV-field of (6.2). The set of field theoretic automorphisms of $K$ that leave $k$ elementwise-fixed and commute with the elements of $\Delta$ forms a group $G$ called the *parameterized Picard-Vessiot group* (PPV-group) or *parameterized differential Galois group* of (6.2). One can show that for any $\sigma \in G$, there exists a matrix $M_\sigma \in \mathrm{GL}_n(k_0)$ such that $\sigma(Z) = (\sigma(z_{i,j})) = ZM_\sigma$. Note that $\partial_x$ applied to an entry of such an $M_\sigma$ is 0 since these entries are elements of $k_0$ but that such an entry need not be constant with respect to the elements of $\Pi$. One may think of these entries as functions of $t$. In [8], the authors show that the map $\sigma \mapsto M_\sigma$ is an isomorphism whose image is furthermore a *linear differential algebraic group*, that is, a group of invertible matrices whose entries satisfies some fixed set of polynomial *differential* equations (with respect to the derivations $\Pi = \{\partial_1, \ldots, \partial_r\}$)

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[1] Picard-Vessiot extensions and the related Picard-Vessiot theory is developed in a fuller generality in [13] and [18] but we shall restrict ourselves to the above context to be concrete.

[2] To be precise, we need $k_0$ to be *differentially closed* with respect to $\Pi$, that is, any system of polynomial differential equations in arbitrary unknowns having a solution in an extension field already has a solution in $k_0$. See [8] for a discussion of differentially closed fields in the context of this Galois theory.

in $n^2$ variables. We say that a set $X \subset \mathrm{GL}_n(k_0)$ is *Kolchin closed* if it is the zero set of such a set of polynomial differential equations. One can show that the Kolchin closed sets form the closed sets of a topology, called the *Kolchin topology* on $\mathrm{GL}_n(k_0)$ (*cf.* [6, 7, 8, 14]). In [8], the authors showed that for parameterized systems of linear differential equations with regular singular points, the parameterized monodromy is Kolchin dense in the PPV-group.

The following result shows how the PPV-group can be used to characterize isomonodromy. As in Section 4, let $\mathcal{P}$ be a simply connected subset of $\mathbb{C}^r$ and $\mathcal{D}$ an open subset of $\mathbb{P}^1(\mathbb{C})$ with $x_0 \in \mathcal{D}$. We assume that $A(t,x)$ in Equation (6.2) is analytic in $\mathcal{D} \times \mathcal{P}$. Assume that $\mathbb{P}^1(\mathbb{C})\backslash\mathcal{D}$ is the union of $m$ disjoint disks $D_i$ and that for each $t \in \mathcal{P}$, Equation (6.2) has a unique singular point in each $D_i$ and that this singular point is a regular singular point. Let $\gamma_i, i = 1, \ldots, m$ be the obvious loops generating $\pi_1(\mathcal{D}, x_0)$. We then have that Equation (6.2) is analytic in $\mathcal{D} \times \mathcal{P}$ and we can speak of monodromy matrices $G_i(t)$ corresponding to analytic continuation of a fundamental solution matrix along $\gamma_i$.

**Proposition 8.** (*cf.* [8], Proposition 5.4) Assume that $\mathcal{D}$, $\mathcal{P}$ and Equation (6.2) are as above. Then this equation is isomonodromic in $\mathcal{D} \times \mathcal{P}'$ for some subset $\mathcal{P}' \subset \mathcal{P}$ if and only if the PPV-group of this equation over $k$ is conjugate to $G(\mathbb{C})$ for some linear algebraic group $G$ defined over $\mathbb{C}$.

## 7. An algebraic condition for projective isomonodromy

We now relate the property of projective isomonodromy to properties of the PPV-group.

**Proposition 9.** Let $k, K, A$, and $G$ be as above. Equation (6.2) is projectively isomonodromic if and only if $G$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0) \subset \mathrm{GL}_n(k_0)$.

*Proof.* As noted above, for parameterized systems of linear differential equations with regular singular points, the parameterized monodromy is Kolchin dense in the PPV-group. The group $\mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0) \subset \mathrm{GL}_n(k_0)$ is the homomorphic image of the linear differential group $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{Scal}_n(k_0) \subset \mathrm{GL}_n(k_0) \times \mathrm{GL}_n(k_0)$ and so by Proposition 7 of [6], it is also Kolchin closed. Therefore, if the monodromy matrices are in $\mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$, then $G \subset \mathrm{Scal}_n(k_0) \cdot \mathrm{GL}_n(\mathbb{C})$. The converse is clear. □

One easy consequence of Propostion 9 is

**Corollary 10.** Let $k, K, A$, and $G$ be as above. If (6.2) is projectively isomonodromic then $(G, G)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{C})$.

This corollary yields a simple test to show that (6.2) is not projectively isomonodromic: If the eigenvalues of the commutators of the monodromy matrices (with respect to any fundamental solution matrix) are not constant, then (6.2) is not projectively isomonodromic. In particular, if the determinant or trace of any of these matrices is not constant then (6.2) is not projectively isomonodromic. The converse of the corollary is not true in general (see Remark 7.1 below) but it is true if Equation (6.2) is absolutely irreducible, that is, when (6.2) does not factor over $\overline{k}$, the algebraic closure of $k$. Before we prove this, we will discuss some group theoretic facts.

In the following, we say that a subgroup $H \subset \mathrm{GL}_n(k_0)$ is *irreducible* if the only $H$-invariant subspaces of $k_0^n$ are $\{0\}$ and $k_0^n$.

**Lemma 11.** Let $H$ be an irreducible subgroup of $\mathrm{GL}_n(\mathbb{C})$ and let $g \in \mathrm{GL}_n(k_0)$ normalize $H$. Then $g \in \mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$.

*Proof.* For any $h \in H$ and $g \in \mathrm{GL}_n(k_0)$ normalizing $H$, we have that

$$0 = \partial_i(g^{-1}hg) = -g^{-1}\partial_i(g)g^{-1}hg + g^{-1}h\partial_i(g)$$

for all $\partial_i \in \Pi$. Therefore,

$$\partial_i(g)g^{-1}h = h\partial_i(g)g^{-1}.$$

Since $H$ is irreducible, Schur's Lemma implies that $\partial_i(g)g^{-1} \in \mathrm{Scal}_n(k_0)$. This means that if $g = (g_{r,s})$, then there exists a $z_i \in k_0$ such that $\partial_i g_{r,s} = z_i g_{r,s}$ for all $r, s$. One can check that the $z_i$ satisfy the integrability conditions so there exists a nonzero $u \in k_0$ such that $\partial_i u = z_i u$ for all $i$. Therefore $g_{r,s} = h_{r,s}u$ for some $h_{r,s} \in \mathbb{C}$ and so $g = uI_n \cdot h$ for some $h \in \mathrm{GL}_n(\mathbb{C})$. □

It is well known that if $G$ and $H$ are linear algebraic groups with $H$ normal in $G$, then $G/H$ is also a linear algebraic group. For $\mathrm{Scal}_n(k_0) \lhd \mathrm{GL}_n(k_0)$, we will denote by $\rho$ the canonical map $\rho : \mathrm{GL}_n(k_0) \to \mathrm{GL}_n(k_0)/\mathrm{Scal}_n(k_0)$.

**Lemma 12.** Let $H \subset \mathrm{GL}_n(k_0)$ be a Kolchin connected linear differential algebraic group and let $\overline{H}$ be its Zarski closure in $\mathrm{GL}_n(k_0)$. Assume that that $\overline{H}$ is irreducible. Then

$$H \subset (H, H)_\Pi \cdot \mathrm{Scal}_n(k_0),$$

where $(H, H)_\Pi$ is the Kolchin closure of $(H, H)$.

*Proof.* Since $\overline{H}$ is irreducible, it must be reductive ([20], p. 37). Since $H$ is Kolchin connected, $\overline{H}$ is Zariski connected so we can write $\overline{H} = Z(\overline{H}) \cdot (\overline{H}, \overline{H})$ where $Z(\overline{H})$ is the center of $\overline{H}$ ([11], Ch. 27.5). Using the irreducibility again, Schur's Lemma implies that $Z(\overline{H}) \subset \mathrm{Scal}_n(k_0)$. Using the map $\rho$ above, we have that $\rho(\overline{H})$ is isomorphic to $(\overline{H}, \overline{H})/(Z(\overline{H}) \cap (\overline{H}, \overline{H}))$ and so is a connected semisimple linear algebraic group. Furthermore, $\rho(H)$ is a Zariski dense, Kolchin connected, subgroup of $\rho(\overline{H})$. Propositions 11 and 13 and of [7] imply that $\rho(H)$ equals $(\rho(H), \rho(H))_\Pi$, the Kolchin closure of its commutator subgroup $(\rho(H), \rho(H))$. Since $(H, H)_\Pi$ is a linear differential algebraic group, we have that $\rho((H, H)_\Pi)$ is a linear differential algebraic group containing $(\rho(H), \rho(H))$ and therefore contains $(\rho(H), \rho(H))_\Pi$. Since $(H, H)_\Pi \subset H$ we that $\rho((H, H)_\Pi) = \rho(H)$. Therefore $H \subset (H, H)_\Pi \cdot \mathrm{Scal}_n(k_0)$. □

**Lemma 13.** Let $G \subset \mathrm{GL}_n(k_0)$ be a linear differential group and assume that

(1) $(G, G) \subset \mathrm{GL}_n(\mathbb{C})$ and
(2) the identity component $\overline{G}^0$ of $\overline{G}$, the Zariski closure of $G$ in $\mathrm{GL}_n(k_0)$, is irreducible.

Then $G \subset \mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$.

*Proof.* We first note that the Zariski closure of $G^0$, the Kolchin component of the identity of $G$ is Zariski connected and of finite index in $\overline{G}$. Therefore $\overline{G}^0$ is the Zariski closure $\overline{G^0}$ of $G^0$. We now apply Lemma 12 to $H = G^0$ and conclude that $G^0 \subset (G^0, G^0)_\Pi \cdot \mathrm{Scal}_n(k_0)$. Since $(G, G) \subset \mathrm{GL}_n(\mathbb{C})$ we have that $(G^0, G^0)_\Pi \subset \mathrm{GL}_n(\mathbb{C})$. Furthermore, since $\overline{G}^0$ is irreducible and is the Zariski closure of $G^0$, we have that $G^0$ is irreducible. Therefore $(G^0, G^0)_\Pi$ is an irreducible subgroup of $\mathrm{GL}_n(\mathbb{C})$. Any $g \in G$ normalizes $G^0$ and therefore normalizes $(G^0, G^0)_\Pi$. Applying Lemma 11 to $H = (G^0, G^0)_\Pi$, we have that $G \subset \mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$. □

*Remark* 7.1. Simple examples (*e.g.*, $G = \mathrm{Diag}_n(k_0)$, the group of diagonal matrices) show that condition $(G, G) \subset \mathrm{GL}_n(\mathbb{C})$ does not imply $G \subset \mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$ without some additional hypotheses.

**Proposition 14.** Let $k, K, A, G$ be as in Proposition 9. If Equation (6.2) is absolutely irreducible and $(G, G)$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{C})$, then (6.2) is projectively isomonodromic.

*Proof.* As noted above, $\overline{G}$ is the usual Picard-Vessiot group of (6.2) over $k$. If (6.2) is absolutely irreducible, then $\overline{G}^0$ is an irreducible subgroup of $\mathrm{GL}_n(k_0)$. Lemma 13 implies that $G \subset \mathrm{GL}_n(\mathbb{C}) \cdot \mathrm{Scal}_n(k_0)$ □

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