Solutions of Algebraic Differential Equations

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I. INTRODUCTION

This paper may be considered as a mathematical essay on the question "What is a solution of an algebraic differential equation?"

Many theorems in differential algebra are proved by differentiating an algebraic differential equation several times, and then eliminating certain quantities, say, by the use of resultants. We give the simplest example of this, that every $C^\infty$ solution of an ADE satisfies some autonomous ADE. For let

$$P(x, y) = 0,$$

where $y = y, y', y'', ..., y^{(n)}$, where $y = y(x)$. Then

$$\frac{d}{dx} P(x, y) = 0.$$

This last expression is a polynomial in $x, y, y', ..., y^{(n+1)}$. Eliminating $x$ from (1) and (2) via resultants (see [1]), we get

$$R_x(y, y', ..., y^{(n+1)}) = 0,$$

which is the desired autonomous ADE. In the general context of such results, it is natural to ask, what if we study solutions that are only $C^n$ instead of $C^\infty$ (or analytic), where $n$ is the order of the ADE? Across the board, we then get negative results. From now on, we shall call a solution in $C^n$ a pointwise solution.

For example, the Ritt–Raudenbush basis theorem, which is one of the most beautiful theorems in mathematics, has as a corollary [4, Theorem 7.4, p. 48] that if $\Sigma$ is an infinite system of ADEs in a finite number of dependent variables, and if $I$ is an open interval on the real axis $\mathbb{R}$, then there is a finite subsystem $\Sigma_f$ with the same $C^\infty$ (or analytic) solutions on $I$ that $\Sigma$ has. In
particular, if $\Sigma$ has no $C^\infty$ (or analytic) solutions, then $\Sigma_f$ also has none. This may be viewed as a "logical compactness" theorem for ADEs (see [2]). Here we construct counterexamples to the corresponding statements for pointwise solutions, which might seem a very natural notion of "solution." (Of independent interest is Lemma 1, which provides a third-order ADE that has no $C^\infty$ solution on a specified interval $I$, but that does have a $C^n$ solution for every $n$.)

In a speculative vein, we note that Shannon [15] and Pour-El [9, 10] have established a strong link between systems of ADEs and analog computers. Therefore, there is a sense in which we have described an analog computer with infinitely many components, such that no subcomputer with only finitely many components has the same outputs as the original computer, contrary to what we might expect from the Ritt–Raudenbush basis theorem. To the extent to which the central nervous system has parts that function as an analog computer (see [12]) our results might have something to do with redundancy in neurophysiology. We shall not dwell on these aspects here.

Finally, it is well known [7, 8] that if $f$ and $g$ satisfy (nontrivial) ADEs, then so do $f+g, f \cdot g, f - g, f/g$, and $f \circ g$ (under suitable restrictions so that the expressions make sense). Closer examination of the proofs reveals that we are really talking about solutions that are at least $C^\infty$, since derivatives of an unspecified order are taken. Indeed, we show that all the corresponding results fail for pointwise solutions. Thus, for example, we find $f$ and $g$ such that each satisfies a nontrivial ADE, but such that $f+g$ does not. Also, we find a $C^1$ function that satisfies an ADE, but that satisfies no ADE with integer coefficients, in sharp contrast to the situation for $C^\infty$ solutions.

Lest the tenor of this paper be overwhelmingly negative, we seek an appropriate weakening of the notion of "solution" of an ADE so that the positive results persist in the new context. We propose that a function $y$ on an open interval $I$ be called an "effective solution" of an ADE, $P(x, y) = 0$, if (i) $y$ is continuous on $I$, (ii) $y$ is analytic on a dense open subset $\Omega$ of $I$, and (iii) $P(x, y) = 0$ for each $x \in \Omega$. It is then easy to carry over the body of the above results from classical differential algebra to this new context.

It turns out that this definitions is less restrictive than we might suppose. Indeed, every pointwise solution is an effective solution. That is, if $y$ is a $C^n$ solution of an ADE, on $I$, then there is a dense open subset of $I$ on which $y$ is analytic. Finally, we show that this result is sharp, by writing down a third-order ADE, $P(x, y) = 0$, such that for every open interval $I$ and every dense open subset $\Omega$ of $I$ there exists a $C^\infty$ solution $y$ of $P(x, y) = 0$, such that $y$ is analytic exactly on $\Omega$—it fails to be analytic at every point of $I \setminus \Omega$.

We thank Lawrence G. Brown for pointing out some errors in the original version of Lemma 2.
II. The Ritt–Raudenbush Basis Theorem

Our first result is relatively easy to establish.

**Theorem II.1.** There exists a countably infinite system $\Sigma$ of algebraic differential equations in one unknown function $y$ so that no finite system $\Sigma_f$ of ordinary differential equations (algebraic or not) has the same pointwise solutions on $\mathbb{R}$ as $\Sigma$.

**Proof.** The equation $(xy'/y)' = 0$ is easily seen to have as its pointwise solutions on $\mathbb{R}$ exactly those twice-differentiable functions $y$ which are locally of the form

$$y = k|x|^\alpha.$$

Looking at the numerator of $(xy'/y)'$, we let

$$P(x, y) = yyy' + xyy'' - xy'^2.$$

Then the same is true of the solutions of $P(x, y) = 0$.

Let $P_n(x, y) = \frac{d^n}{dx^n} P(x, y)$, so that $P_n$ involves $x, y, y', \ldots, y^{(n+2)}$. Let $\Sigma$ consist of the equations $P_n(x, y) = 0$ for $n = 0, 1, 2, \ldots$. For $y$ to be a solution of $\Sigma$ (even to be able to plug $y$ into $\Sigma$ meaningfully), we must have $y \in C^\infty$. It follows that the solutions on $\mathbb{R}$ of $\Sigma$ are exactly $y = kx^n$, $k \in \mathbb{R}$, $n = 0, 1, 2, \ldots$.

Now let $\Sigma_f$ be any finite system of ODEs that has every $y = kx^n$ as a solution, and let $N$ be the maximum order of differentiation that enters into $\Sigma_f$. Let

$$y_N(x) = \begin{cases} 0, & x \leq 0, \\
-x^{N+3}, & x > 0. \end{cases}$$

Then $y_N \in C^N$ is differentiable enough to be a pointwise solution of $\Sigma_f$. And indeed it does satisfy every equation in $\Sigma_f$ because it does so for $x \leq 0$ and $x > 0$. On the other hand, $y_N$ is not a solution of $\Sigma$.

**Theorem II.2.** On a certain open interval $I$, say $I = (-\pi/2, (\pi/2) + x_0)$, where $0 < x_0 < \pi/2$, there exists an infinite system $\Sigma$ of algebraic differential equations that has no (pointwise) solution on $I$, yet every finite subsystem $\Sigma_f$ has a pointwise solution on $I$.

This result is somewhat harder to prove than Theorem 1. It gives a counterexample to "logical compactness" (see [2, Theorem 1.3.22, p. 33]) for differential equations.
The construction of $\Sigma$ follows easily from

**Lemma 1.** There exists an ADE, $P(x, y, y', y'', y''') = 0$, of the third order, that has no $C^\infty$ solution on $I$, but that has a $C^n$ solution for every $n$.

Given Lemma 1, it is easy to construct $\Sigma$ as above, by letting $P_n(x, y) = (d^n/dx^n) P(x, y)$, and proceeding as above. The proof of Lemma 1 follows from

**Lemma 2.** Consider the algebraic differential equation

$$x(1 + x)^2 (zz' + x(zz'' - z')^2) = (\frac{1}{4}z^2 - 2xzz' - 2x^2(zz'' - z')^2)^2. \quad (*)$$

The solutions are $z = 0$ for $x \leq 0$, and for $x > 0$ either $z = k(x^a + x^{a+1/2})$ or $z = kx^a \exp \frac{1}{2} \int x^{-1}(1 - x^{1/2})^{-1} dx$, where $k$ and $a$ are arbitrary real constants. The only solution that is $C^\infty$ in a two-sided neighborhood of $x = 0$ is $z = 0$.

**Proof.** The equation is derived by taking $z = k(x^a + x^{a+1/2})$, or $v = z/(1 + x^{1/2}) = kx^a$ and eliminating $k$ and $a$ through $((xv')/v)' = 0$. In the process, the $x^{1/2}$ has to be eliminated by squaring. The idea is that this derivation may be reversed, but the other square root leads to the other solution. Because of the possibility of error in this kind of argument, we provide a lot of details.

First of all, if $x < 0$, then both sides of $(*)$ must be zero since otherwise the left side would be negative while the right side, a square. Hence $zz' + x(zz'' - z')^2 = 0$, $\frac{1}{4}z^2 - 2x(zz' + x(zz'' - z')^2) = 0$, and thus $z = 0$. Let us now suppose $z \neq 0$. Then $zz' + x(zz'' - z')^2 \neq 0$ as we have seen. Dividing, we get

$$x + 2x^2 + x^3 = \left[ \frac{2xzz' + 2x^2(zz'' - z')^2 - z^2/4}{zz' + x(zz'' - z')^2} \right]^2. \quad (#)$$

Let us first take one square root in $(#)$. Later we will take the other.

$$x^{1/2} + x^{3/2} = -\frac{2xzz' + 2x^2(zz'' - z')^2 - z^2/4}{zz' + x(zz'' - z')^2},$$

$$zz'(x^{1/2} + x^{3/2} + 2x) + x(x^{1/2} + x^{3/2} + 2x)(zz'' - z') - \frac{1}{4}z^2 = 0,$$

$$zz'(1 + 2x^{1/2} + x) + x(1 + 2x^{1/2} + x)(zz'' - z') - \frac{1}{4}x^{-1/2}z^2 = 0,$$

on dividing by $x^{1/2}$.

$$zz'(1 + x^{1/2})^2 + x(1 + x^{1/2})^2 (zz'' - z') - \frac{1}{4}x^{-1/2}z^2 = 0.$$
Divide by \( z^2(1 + x^{1/2})^2 \),

\[
\frac{z'}{z} + x \frac{zz'' - z'}{z^2} - \frac{1}{4} \frac{x^{-1/2}}{(1 + x^{1/2})^2} = 0,
\]

\[
\frac{d}{dx} \left( x \frac{z'}{z} \right) - \frac{1}{2} \frac{x^{-1/2}/2}{(1 + x^{1/2})^2} = 0.
\]

Hence

\[
\frac{xz'}{z} + \frac{(1/2)}{1 + x^{1/2}} = c, \quad \frac{xz'}{z} = \frac{x^{1/2}}{1 + x^{1/2}} + C,
\]

\[
\frac{z'}{z} = \frac{x^{-1/2}/2}{1 + x^{1/2}} + \frac{C}{x} , \quad \log |z| = \log(1 + x^{1/2}) + C \log x + d
\]

or

\[
z = kx^a (1 + x^{1/2})
\]

as desired. It is easy enough, say by reversing these steps, to see that (3) is a solution of (*).

Now for the other square root—the calculations are similar.

\[-(x^{1/2} + x^{3/2}) = - \frac{2xzz' + 2x^2(zz'' - z'^2) - z^2/4}{zz' + x(zz'' - z'^2)},\]

\[
zz'(-(x^{1/2} + x^{3/2}) + 2x) + x(-(x^{1/2} + x^{3/2}) + 2x)(zz'' - z'^2) - \frac{1}{4}z^2 = 0,
\]

\[
z'(-1 + x) + 2x^{1/2} + x(-(1 + x) + 2x^{1/2})(zz'' - z'^2) - \frac{1}{4}z^2x^{-1/2} = 0,
\]

\[
z'(-1 - x^{1/2})^2 + x(1 - x^{1/2})^2 (zz'' - z'^2) + \frac{1}{4}z^2x^{-1/2} = 0.
\]

Divide by \( z^2(1 - x^{1/2})^2 \) (supposing \( x \neq -1 \)—the case \( x = -1 \) is handled by continuity),

\[
\frac{z'}{z} + x \frac{zz'' - z'^2}{z^2} + \frac{1}{4} \frac{x^{-1/2}}{(1 - x^{1/2})^2} = 0,
\]

\[
\frac{d}{dx} \left( x \frac{z'}{z} \right) - \frac{1}{2} \frac{d}{dx} \frac{1}{1 - x^{1/2}} = 0
\]

\[
x \frac{z'}{z} = \frac{1}{2} \frac{1}{1 - x^{1/2}} + C,
\]

\[
\frac{z'}{z} = \frac{1}{2}x \frac{1}{1 - x^{1/2}} + \frac{C}{x}
\]

\[
\log |z| - C \log x + \int \frac{1}{2x(1 - x^{1/2})} dx.
\]
As desired

\[ z = kx^\alpha \exp \int \frac{1}{2x(1 - x^{1/2})} \, dx \quad (4) \]

and it may be verified that (4) also is a solution of (*).

Now to see that (*) has no \( C^\infty \) solutions other than \( z \equiv 0 \), in a two-sided neighborhood of \( x = 0 \), we note that \( z \equiv 0 \) for \( x < 0 \), so that for \( x > 0 \), \( z = k(x^\alpha + x^{\alpha + 1/2}) \) is impossible unless \( k = 0 \). One way to see this is to prove by l'Hospital's rule that if \( z \in C^\infty \) to the right of \( x = 0 \) and if \( z^{(l)}(0) = 0 \) for all \( l = 0, 1, 2, \ldots \), then \( z/x^\alpha = o(1) \) for all real \( \alpha \). But \( z/x^\alpha = k(1 + x^{1/2}) \) so we must have \( k = 0 \). Similarly for the solution (4), rewrite it as

\[ z = kx^d \exp \int \left[ \frac{1}{2x(1 - x^{1/2})} - \frac{1}{2x} \right] \, dx = kx^d \exp \int \frac{x^{1/2}}{2x(1 - x^{1/2})} \, dx \]

and repeat the last argument. The result is proved.

**Proof of Lemma 1.** In the ADE of Lemma 2, substitute \( y' - (y^2 + 1) \) for \( z \). If \( y \) were \( C^\infty \), so would be \( y' - (y^2 + 1) \), so that we must have \( y' - (y^2 + 1) = 0 \) or \( y = \tan(x + d) \), which blows up in \( I \) because our interval \( I \) has length exceeding \( \pi \). To construct a \( C^n \) solution for any given \( n \), we let \( y = \tan x \) for \(-\pi/2 < x \leq 0\). Now we must worry about the interval \([0, \pi/2 + x_0]\). We note that \( x_0^\alpha + x_0^{\alpha + 1/2} > 0 \), and we choose \( a \) so large that \( z(x) = k(x^\alpha + x^{\alpha + 1/2}) \) belongs to \( C^n(I) \). This is easy to arrange because of the order of vanishing of \( x^\alpha \) at \( x = 0^+ \). We will choose \( k = -K \), where \( K \) is a large positive number. Let \( y_K \) be the unique solution on a suitable interval (of the form \([0, S]\)) of the initial value problem

\[ y' - (y^2 + 1) = -K(x^\alpha + x^{\alpha + 1/2}), \quad y(0) = 0. \quad (5) \]

Of course it is conceivable that \( y_k \) goes to \( +\infty \) in a finite interval \([0, t]\), \( 0 < t < (\pi/2) + x_0 \). We will see that this cannot happen if \( K \) is large enough. We observe that

\[ -Kg(x) \leq y_K(x) \leq y_0(x) = \tan x, \]

where

\[ g(x) = \int_0^x (r^\alpha + r^{\alpha + 1/2}) \, dr. \]

Now choose \( x_1 \) with \( x_0 < x_1 < \pi/2 \). In particular

\[ \delta = \min\{x^\alpha + x^{\alpha + 1/2} : x_0 \leq x \leq x_1\} > 0. \]
We will show that, for \( K \) sufficiently large, there exists an \( x' \) with
\( x_0 \leq x' < x_1 \) so that \( y_K(x') \leq 0 \). Then we are done, for \( y'_K(x) \leq y_K(x)^2 + 1 \) and
\( y'_K(x') \leq 0 \) so that \( y_K(x) \leq \tan(x - x') \leq \tan(x - x_0) \), which remains finite up to
\( x = ((\pi/2) + x_0)^- \). Since also \( y_K(x) \geq -Kg(x) \), our function \( y_K \) will stay finite over \( I \). Now to show that there exists such a \( K \). We suppose, on the contrary, that \( y_K(x) > 0 \) for \( x_0 < x < x_1 \). We have
\[
y_K(x) < y_K(x') + 1 - K\delta, \quad x_0 < x < x_1
\]
so that
\[
y_K(x) \leq \tan^2 x_1 + 1 - K\delta \quad \text{for} \quad x_0 < x \leq x_1
\]
since \( 0 < y_K(x) \leq \tan x_1 \) in that interval. For a given number \( N \), choose \( K \) so large that the right side is less than \(-N\). Then \( y_K(x_1) \leq y_K(x_0) - N(x_1 - x_0) \leq \tan x_0 - N(x_1 - x_0) \), and on choosing \( N > (x_1 - x_0)^{-1} \tan x_0 \) we would have
\( y_K(x_1) < 0 \). This contradiction shows that \( y_K(x') < 0 \) for some \( x' \in [x_0, x_1] \) in any event.

We describe an open question. Let us say that a differential polynomial
\( P(x, y) \) is irreducible if
\[
P = \sum F_\alpha Q^{(\alpha)} = \sum F_\alpha Q^{(\alpha)}
\]
implies that either \( Q = aP \) or \( Q = a \), where \( a \) is a generic constant.

**Question.** If \( \Sigma \) is an infinite system of irreducible algebraic differential equations, must there exist a finite subsystem \( \Sigma_f \) with the same effective solutions as \( \Sigma \)?

**Remark.** It is now easy to prove the next result, whose proof we only sketch.

**Theorem II.3.** There exists a second-order algebraic differential equation on an open interval \( I \) that has \( C^\infty \) solutions on \( I \) but has no analytic solutions on \( I \).

**Sketch of proof.** Let \( I = (-\pi/2, (\pi/2) + x_0) \) and study \( y' - (y^2 + 1) = ke^{-1/x} \). Note that \( z = ke^{-1/x} \) are exactly the solutions of \( x^2 z' - z = 0 \). So the desired ADE reads \( x^2 [y'' - 2yy'] - [y' - (y^2 + 1)] = 0 \).

**III. THE FAILURE OF SOME OTHER CLASSICAL RESULTS IN THE CONTEXT OF POINTWISE SOLUTIONS**

**Theorem III.1.** The ADE
\[
y'^2 = x^2 4y(1 - y)
\]
has a pointwise solution that satisfies no autonomous ADE
\[
Q(y, y', ..., y^{(n)}) = 0.
\]
Proof. Let us piece together a solution $y$ that is flat zero except on intervals $I_k = \{a_k, \sqrt{2\pi + a_k^2}\}$, where, if $b_k = a_k^2/2$, then $\{b_k\}$ has a limit point modulo $2\pi$, but $a_k \to \infty$ very fast. On $I_k$, let

$$y = \sin^2 \left( \frac{x^2 - a_k^2}{2} \right).$$

It is easy to check that $y$ is a $C^1$ solution of the first-order equation ($\ast$). Note, however, that $y$ does not have a well-defined second derivative at many points, so that if ($\neq$) holds, then $n = 1$ must hold. Thus

$$Q \left( \sin^2 \left( \frac{x^2 - a_k^2}{2} \right), 2x \sin \left( \frac{x^2 - a_k^2}{2} \right) \cos \left( \frac{x^2 - a_k^2}{2} \right) \right) = 0 \quad (\dagger)$$

for $x \in I_k$. But the left-hand side of ($\dagger$) is an analytic function of $x$, so that ($\dagger$) must hold for all $x$, for $k = 1, 2, 3, \ldots$. But again by the analyticity, since $\{a_k^2/2\}$ has a limit point modulo $2\pi$, we must have

$$Q \left( \sin^2 \left( \frac{x^2 - a^2}{2} \right), 2x \sin \frac{x^2 - a^2}{2} \cos \frac{x^2 - a^2}{2} \right) = 0$$

for all real $x$ and $a$.

But on taking the Jacobian and applying the implicit function theorem, we see that there is an open set $E \subseteq \mathbb{R}^2$ so that for all $(A, B) \in E$, we may find $x$ and $a$ in $\mathbb{R}$ so that

$$\sin^2 \left( \frac{x^2 - a^2}{2} \right) = A, \quad 2x \sin \frac{x^2 - a^2}{2} \cos \frac{x^2 - a^2}{2} = B.$$ 

Thus we have $Q(A, B) = 0$ for all $(A, B) \in E$ and hence $Q$ is the trivial polynomial, which proves our assertion.

**Theorem III.2.** Consider the ADE

$$y' = 4y(1 - y).$$

There exist two $C^1$ solutions $y_1$ and $y_2$ of this equation such that $y_1 + y_2$ satisfies no ADE at all.

Remark. Simple modifications of the proof will handle $f \cdot g, f - g, f/g,$ and $f \circ g$.

**Proof of Theorem III.2.** The proof goes somewhat like that of the preceding result, so we will be brief. Choose a sequence $(a_n, b_n)$ with $a_n \to \infty, b_n \to \infty$ both rather fast, so that $\{(a_n, b_n)\}$ is dense when reduced modulo $2\pi$ in both variables to $[0, 2\pi] \times [0, 2\pi]$. Let $y'_1(x)$ be zero except on
intervals \( I^1_n = (a_n, a_n + 2\pi) \) and let \( y_1(x) \) be zero except on intervals \( I^2_n = (b_n, b_n + 2\pi) \), and suppose further that for each \( n \) \( I^1_n \cap I^2_n \) is not empty. On \( I^1_n \) let \( y_1 = \sin^2(x - a_n) \) and on \( I^2_n \) let \( y_2 = \sin^2(x - b_n) \). It is easy to see that \( y = y_1 + y_2 \) has no second derivative at many points, so if \( y \) satisfies an ADE, \( Q = 0 \), then \( Q \) must be first order

\[
Q(x, y(x), y'(x)) = 0.
\]

We have

\[
Q(x, \sin^2(x - a_n) + \sin^2(x - b_n), 2 \sin(x - a_n) \cos(x - a_n) + 2 \sin(x - b_n) \cos(x - b_n)) = 0
\]

for \( x \in I^1_n \), and so by analyticity for all \( x \in \mathbb{R} \). Then since \((a_n, b_n)\) is dense mod \( 2\pi \times 2\pi \), we must have

\[
Q(x, \sin^2(x - a) + \sin^2(x - b), 2 \sin(x - a) \cos(x - a) + 2 \sin(x - b) \cos(x - b)) = 0
\]

for all \( x, a, b \in \mathbb{R} \). Consider the equations

\[
\sin^2(x - a) + \sin^2(x - b) = A,
\]

\[
2 \sin(x - a) \cos(x - a) + 2 \sin(x - b) \cos(x - b) = B.
\]

Given any \( x \), there is an open set \( E_x \) of \((A, B)\) in \( \mathbb{R}^2 \) so that these equations hold. Hence we have

\[
Q(x, A, B) = 0 \quad \text{for all} \quad (A, B) \in E_x.
\]

It follows then that

\[
Q(x, A, B) = 0 \quad \text{for all} \quad (A, B) \in \mathbb{R}^2
\]

and thus that \( Q \) is the trivial polynomial, which was to be proved.

The next result contrasts with a result proved for \( C^\infty \) functions by Ritt and Gourin [13] (see also [6]).

**Theorem III.3.** There exists a \( C^1 \) function \( y \) that satisfies an algebraic differential equation, but that satisfies no algebraic differential equation with integer coefficients.

**Proof.** Let \( y = \pi \sin^2(x - a_k) \) on disjoint intervals \( I_k \) of length \( 2\pi \) centered at \( a_k \), with \( y = 0 \) otherwise, where the \( a_k \) are dense modulo \( 2\pi \). This function satisfies the ADE \( y'^2 - 4y(\pi - y) = 0 \). We now show that it satisfies no ADE \( P(x, y) = 0 \), where \( P \) has integer coefficients. Since there are points
where \( y'' \) fails to exist, we see that this equation is of order 1, i.e., \( P(x, y, y') = 0 \). As in earlier arguments, we must have \( P(x, \pi \sin^2(x - a), 2\pi \sin(x - a) \cos(x - a)) = 0, \forall x, a \). If \( P \) is not identically zero, then there must exist a rational number \( r \) so that \( P(r, s, t) \) is not the zero polynomial in \( s \) and \( t \).

But note that \( P(r, s, t) = 0 \) whenever \( t = \pm 2 \sqrt{s \sqrt{\pi - s}}, 0 \leq s \leq \pi \), because we may then take \( s = \pi \sin^2(x - a) \), etc. It follows that \( P(r, s, t) \) is divisible by \( t^2 - 4s(\pi - s) \). Hence \( Q(s) = P(r, s, s) \) is divisible by \( s^2 - 4s(\pi - s) = s(5s - 4\pi) \). Hence \( Q(4\pi/5) = 0 \), which is impossible since \( 4\pi/5 \) is transcendental, yet \( Q \) has rational coefficients.

IV. Some Positive Results

**Theorem IV.1.** Let \( y \) be a \( C^n \) solution of an ADE

\[
P(x, y) = 0
\]

on an open interval \( I \). Then there is a dense open subset \( \Omega \) of \( I \) on which \( y \) is analytic.

**Remark.** This result is apparently asserted in [10] just at the end of the proof of Theorem 2, but no proof is offered.

**Proof.** The proof is by induction on the order of \( P \) first, and then on the degree of \( P \) within a given order. The result is clearly true for order 0 and degree 1, the case where \( y \) is a rational function, \( ay + b = 0 \), so \( y = -b/a \), and the zeros of \( a \) are isolated. Now let \( S \) be the "separant"

\[
S = \frac{\partial P}{\partial y^{(n)}}.
\]

Then \( S \) is lower than \( P \), i.e., either has the same order and lower degree or has lower order. Let \( \Sigma = \{ x \in I : S(x, y(x)) \neq 0 \} \) and let \( E = I \setminus \Sigma \). For \( x \in \Sigma \), we can, by the implicit function theorem, solve for

\[
y^{(n)}(x) = A(x, y(x), \ldots, y^{(n-1)}(x))
\]

in a neighborhood of \( x \), where \( A \) is an analytic function of its variables. Since \( y \in C^n \), \( \Sigma \) is an open set. By the fundamental existence and uniqueness theorem [3], \( y \) is analytic in \( \Sigma \). This leaves \( E \) to worry about. But on \( E^0 \), the interior of \( E \), \( y \) satisfies the lower ADE, \( S = 0 \), so that by induction, \( y \) is analytic on a dense open subset of \( E^0 \). And \( \partial E = E \setminus E^0 \) is nowhere dense. This completes the proof.
Theorem IV.2. There exists a third-order ADE \((*)\) \(P(x, y) = 0\) so that given any dense open subset \(\Omega\) of \(I\), there is a \(C^\infty\) solution \(y\) of \((*)\) that is analytic on \(\Omega\) and nonanalytic at every point of \(E = I \setminus \Omega\).

Proof. Write \(\Omega\) as a disjoint union of countably many open intervals \((a_n, b_n), n = 1, 2, \ldots\). (The case of finitely many intervals \((a_n, b_n)\) is even easier to handle.) Let

\[
f(x) = e^{-1/(1-x^2)}, \quad |x| < 1, \\
= 0, \quad |x| \geq 1,
\]

and let \(f_n(x) = f(A_n x + B_n)\), where \(A_n a_n + B_n = -1, A_n b_n + B_n = +1\). As in [14], we find a third-order ADE

\[P(x, y, y', y'', y''') = 0\]

such that every function of the form \(y = kf'_n\), where \(k\) is a constant, is a solution. Now we let

\[y = \sum k_n f_n, \tag{\#}\]

and claim that if \(k_n \to 0\) sufficiently fast, then \(y\) has the properties claimed. Notice that at most one summand in \((\#)\) is nonzero at a given \(x\), so there is no question of the meaning of \((\#)\). Also, it is clear that \(y\) is analytic in \(\Omega\). We shall prove that \(y\) and all its derivatives exist and vanish at each point of \(E\), if \(k_n \to 0\) fast enough. We sketch the proof here. One must consider left-hand derivatives \(D_-\) and right-hand derivatives \(D_+\). First of all, if \(k_n\) merely approaches 0, then \(y\) is continuous. If \(x\) is a left-hand endpoint of an interval \(I_n\), then clearly \((D_-^k y)(x) = 0\) for each \(k = 0, 1, 2, \ldots\); similarly if \(y\) is a right-hand endpoint. Now suppose we already have \((D_-^k y)(x) = 0\) and want to prove \((D_+^{k+1} y)(x) = 0\), where \(x\) is not a left-hand endpoint of any \(I_n\). Let \(I_n\) lie to the right of \(x\). We must consider

\[
\frac{y^{(k)}(z) - y^{(k)}(x)}{z - x} \quad \text{for} \quad z \in I_n, \quad \text{so that} \quad y^{(k)}(a_n) = y^{(k)}(x) = 0.
\]

We have

\[
\frac{y^{(k)}(z) - y^{(k)}(x)}{z - x} = \frac{y^{(k)}(z) - y^{(k)}(a_n)}{z - x} \leq \frac{y^{(k)}(z) - y^{(k)}(a_n)}{a_n x} \leq k_n f_n^{(k+1)}(\xi),
\]

where \(a_n \leq \xi \leq z\). Thus we choose \(k_n\) so that

\[
k_n \max \{|f_n^{(j)}(\xi)|: a_n \leq \xi \leq b_n, j = 1, \ldots, n\} = \frac{1}{n}.
\]
say, and it follows that $D_n^k(x) = 0$ for $k = 0, 1, 2, \ldots$, since $x$ is the limit of a decreasing sequence of $a_n$, so that $n \to \infty$. This completes the proof of the theorem.

REFERENCES