Galois Theory of Parameterized Differential Equations

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Usual Picard-Vessiot Theory

Parameterized Picard-Vessiot Theory

Integrable and Isomonodromic Families

$2 \times 2$ Parameterized Families
Algebraic Formalism:

\( k = \) a differential field of characteristic zero with derivation \( \partial \)

\( k_\partial = \{ c \in k \mid \partial c = 0 \} \), the \( \partial \)-constant subfield

\[ \partial Y = AY, \quad A \in M_n(k) \]
Algebraic Formalism:

$k$ = a differential field of characteristic zero with derivation $\partial$

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$$\partial Y = AY, \ A \in M_n(k)$$

Theorem. 1. If $k_\partial$ is algebraically closed, then there exists a unique field $K$, the PV-extension, such that

- $K = k(z_{1,1}, \ldots, z_{n,n})$ for $Z = (z_{i,j}) \in \text{GL}_n(K)$ with $\partial Z = AZ$, and
- $K_\partial = k_\partial$. 

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2. The PV-group \( \text{Gal}_\partial(K/k) \) of differential \( k \)–automorphisms of \( K \) is a linear algebraic group.
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3. **Galois Correspondence:** \( H^{\text{Zariski closed}} \subset \text{Gal}_\partial(K/k) \iff F^{\text{Diff. field, } k \subset F \subset K} \)
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4. Solvable by liouvillian functions \( \iff \text{Gal}_\partial(K/k) \text{ solvable by finite.} \)
Algebraic Formalism:

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5. If \( \frac{dY}{dx} = AY \) has only regular singular points, then

\[ \text{Monodromy}_{\text{Zariski}} = \text{Gal}_{\partial}(K/k) \]
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$k = \text{a differential field of characteristic zero with derivation } \partial$

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6. $\text{tr.deg.}_k K = \dim \text{Gal}_\partial(K/k)$
A similar result is true for partial differential fields $k$ with derivations $\Delta = \{\partial_1, \ldots, \partial_n\}$, $\partial_i \partial_j = \partial_j \partial_i$, and integrable systems

$$\partial_i Y = A_i Y, \ A_i \in M_n(k), \ i = 1, \ldots n$$

$$[A_j, A_i] = \partial_i A_j - \partial_j A_i$$
A similar result is true for *partial* differential fields \( k \) with derivations \( \Delta = \{ \partial_1, \ldots, \partial_n \} \), \( \partial_i \partial_j = \partial_j \partial_i \), and integrable systems

\[
\partial_i Y = A_i Y, \ A_i \in M_n(k), \ i = 1, \ldots n
\]

\[
[A_j, A_i] = \partial_i A_j - \partial_j A_i
\]

**Ex. 1:**

\[
\frac{dy}{dx} = \frac{1}{2x}y
\]

\( k = \mathbb{C}(x), \ K = k(x^{\frac{1}{2}}), \ Gal_\partial(K/k) = \mathbb{Z}/2\mathbb{Z} \subset GL_1(\mathbb{C}) \)
A similar result is true for partial differential fields $k$ with derivations $\Delta = \{\partial_1, \ldots, \partial_n\}$, $\partial_i \partial_j = \partial_j \partial_i$, and integrable systems

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Ex. 2:

\[
\frac{dy}{dx} = \frac{\sqrt{2}}{x}y
\]

$k = \mathbb{C}(x), \quad K = k(x^{\sqrt{2}}), \quad \text{Gal}_\partial(K/k) = \mathbb{C}^* = \text{GL}_1(\mathbb{C})$

In general,

\[
\frac{dy}{dx} = \frac{t}{x}y, \quad \text{Gal}_\partial(K/k) = \begin{cases} \mathbb{Z}/q\mathbb{Z} & \text{if } t = p/q, (p, q) = 1 \\ \mathbb{C}^* = \text{GL}_1(\mathbb{C}) & \text{if } t \notin \mathbb{Q} \end{cases}
\]
Ex. 3: Bessel equation

\[ \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + (1 - \frac{\nu^2}{x^2})y = 0 \]

or

\[ Y' = AY, \quad A = \begin{pmatrix} 0 & 1 \\ \frac{\nu^2}{x^2} - 1 & -\frac{1}{x} \end{pmatrix} \]

\[ k = \mathbb{C}(x), \quad K = k(J_\nu(x), Y_\nu(x), J'_\nu(x)) \]

If \( \nu - \frac{1}{2} \notin \mathbb{Z} \), then \( \text{Gal}_\partial(K/k) = \text{SL}_2(\mathbb{C}) \)

\[ \downarrow \]

\( J_\nu(x), Y_\nu(x), J'_\nu(x) \) are algebraically independent over \( \mathbb{C}(x) \).
Parameterized Picard-Vessiot Theory

\[
\frac{\partial Y}{\partial x} = A(x, t_1, \ldots, t_m)Y \quad A \in M_n(\mathbb{C}(x, t_1, \ldots, t_m))
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Galois group = the group of transformations of $Y$ that preserve all algebraic relations among $x, t_1, \ldots, t_m, Y$, and the derivatives of $Y$ (with respect to all the variables).
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Formally: \( Y = (y_{i,j}(x, t_1, \ldots, t_n)) \), \( y_{i,j} \) analytic near \( x = x_0, t_1 = \tau_1, \ldots, t_m = \tau_m \)

\( k = \mathbb{C}(x, t_1, \ldots, t_m) \) is a differential field with respect to

\[ \Delta = \{ \partial_0, \partial_1, \ldots, \partial_m \}, \partial_0 = \frac{\partial}{\partial x}, \partial_i = \frac{\partial}{\partial t_i} i = 1, \ldots m. \]

\[ K = k(y_{1,1}, \ldots, y_{n,n}, \ldots, \partial_1^{n_1} \partial_2^{n_2} \cdots \partial_m^{n_m} y_{i,j}, \ldots) \]

= Parameterized Picard-Vessiot (PPV) Extension of \( k \).

\( K \) is a \( \Delta \)–differential field.
Parameterized Picard-Vessiot Theory

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= Parameterized Picard-Vessiot (PPV) Extension of $k$.

$K$ is a $\Delta$-differential field.

The PPV-Group $\text{Gal}_\Delta(K/k)$ is the group of $k$-automorphisms $\sigma$ of $K$ that commute with all $\partial_i, \ i = 0, \ldots, m$.

What do these look like?
Ex. 4:

\[ \frac{\partial y}{\partial x} = \frac{t}{x} y \]

\[ k = \mathbb{C}(x, t), \quad \Delta = \{\partial_x, \partial_t\} \]

\[ K = \mathbb{C}(x, t, x^t, \log x) \]
Ex. 4:

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Let \(\sigma \in \text{Gal}_\Delta(K/k)\).

**Claim 1:** \(\sigma(x^t) = ax^t, \ a \in K, \ \partial_x(a) = 0\) (and so \(a \in \mathbb{C}(t)\))

**Proof:** \(\partial_x \sigma(x^t) = \frac{t}{x} \sigma(x^t)\)
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**Claim 2:** \( \sigma(\log x) = \log x + c, \ c \in \mathbb{C} \)

**Proof:** \( \partial_x \sigma(\log x) = \frac{1}{x} \Rightarrow \sigma(\log x) = \log x + c, \ \partial_x c = 0 \)

\( \partial_t \sigma(\log x) = 0 \Rightarrow \partial_t c = 0 \)
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**Claim 3:** \( \sigma(x^t) = ax^t, \ \partial_x a = 0, \ \partial_t a = ca, \ c \in \mathbb{C} \) (i.e., \( a = e^{ct+d} \))

**Proof:** \( \sigma(\partial_t x^t) = \sigma((\log x + c)ax^t) = a \log x x^t + ca x^t \)

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\( \partial_t \sigma(x^t) = \partial_t(ax^t) = (\partial_t a)x^t + a \log x \ x^t \)

\[ \text{Gal}_\Delta(K/k) = \{ a \in \mathbb{C}(t) \mid \partial_x a = 0, \ \partial_t(\frac{\partial_t a}{a}) = 0 \} \]
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\frac{\partial y}{\partial x} = \frac{t}{x} y
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\[k = \mathbb{C}(x, t), \quad \Delta = \{\partial_x, \partial_t\}\]

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\[\text{Gal}_\Delta(K/k) = \{a \in K^* \mid \partial_x a = 0, \partial_t \left( \frac{\partial_t a}{a} \right) = 0\}\]

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- Let \(k_0 = \ker \partial_x = \mathbb{C}(t)\). This is a \(\partial_t\)-field. The PPV-group is a subgroup of \(\text{GL}_1(k_0)\) defined by differential equations, that is, it is a linear differential algebraic group.
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- For this \( k \) and \( K \),

\[
\text{Gal}_\Delta(K/k) = \{a = e^{ct + d} \in \mathbb{C}(t) \mid c, d \in \mathbb{C}\}
\]

\[
= \mathbb{C}^* = \text{GL}_1(\mathbb{C})!!
\]
\[
\frac{\partial y}{\partial x} = \frac{t}{x} y
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\( k = \mathbb{C}(x, t), \quad \Delta = \{ \partial_x, \partial_t \} \)

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- The function \( e^{2\pi it} \) parametrizes the monodromy of the differential equation and satisfies \( \partial_t \left( \frac{\partial_t a}{a} \right) = 0 \) but is not in \( \text{Gal}_\Delta(K/k) \)!!!
\[
\frac{\partial y}{\partial x} = \frac{t}{x} y
\]

\[k = \mathbb{C}(x, t), \quad \Delta = \{\partial_x, \partial_t\}\]

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\]

- Let \(k_0 = \ker \partial_x = \mathbb{C}(t)\). This is a \(\partial_t\)-field. The PPV-group is a subgroup of \(GL_1(k_0)\) defined by differential equations, that is, it is a linear differential algebraic group.

- For this \(k\) and \(K\),

\[
\text{Gal}_\Delta(K/k) = \{ a = e^{ct+d} \in \mathbb{C}(t) | c, d \in \mathbb{C} \}
\]

\[
= \mathbb{C}^* = GL_1(\mathbb{C})
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- The function \(e^{2\pi i t}\) parametrizes the monodromy of the differential equation and satisfies \(\partial_t \left( \frac{\partial_t a}{a} \right) = 0\) but is not in \(\text{Gal}_\Delta(K/k)\)!

For these (and other) reasons, we need a larger field of \(\partial_x\)-constants or need to consider \(\text{Isom}_\Delta(K, U)\), where \(U\) is some big field or . . .
Definition: Let $k$ be a differential field with derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$. We say that $k$ is differentially closed if any system of polynomial differential equations $f_1(y_1, \ldots, y_n) = \ldots = f_r(y_1, \ldots, y_n) = 0$ with coefficients in $k$ that has a solution in some $\Delta$—differential extension field has a solution in $k$.

- Similar to algebraically closed, real closed, …
- Not hard to construct: Zorn’s Lemma plus …
- Also called
  - constrained closed - Kolchin
  - existentially closed - Robinson, Blum, Shelah, McGrail, Scanlon, Pillay, Pierce, Yaffe, …
Ex. 4(bis): Let \( k_0, \mathbb{C}(t) \subset k_0 \), be a differentially closed \( \partial_t \)-differential field and let \( k = k_0(x) \) be the \( \Delta = \{ \partial_x, \partial_t \} \)-differential field where \( \partial_x(k_0) = 0, \partial_x(x) = 1, \partial_t(x) = 0 \). The differential equation

\[
\partial_x y = \frac{t}{x} y
\]

has Parameterized Picard-Vessiot (PPV) extension

\[
K = k < x^t >_\Delta = k(x^t, \log x)
\]

and has PPV-group

\[
\text{Gal}_\Delta(K/k) = \{ c \in k_0 \mid \partial_t \left( \frac{\partial_t c}{c} \right) = 0 \}.
\]

The parameterized monodromy

\[
m(t) = e^{2\pi it}
\]

generates a subgroup \( \{ e^{2\pi int} \}_{n \in \mathbb{Z}} \) that is Kolchin dense in the PPV-group, that is, any differential polynomial (in \( t \) and \( \partial_t \)) vanishing of this subgroup vanishes on the PPV-group.
Parameterized Picard-Vessiot Theory (redux)

$k$ is a differential field of characteristic zero with derivations $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$

$k_0 = \{c \in k \mid \partial_0 c = 0\}$, the $\Delta' = \{\partial_1, \ldots, \partial_m\}$ field of $\partial_0$-constants

$$\partial_0 Y = AY, \ A \in \text{M}_n(k)$$
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$$\partial_0 Y = AY, \ A \in M_n(k)$$

**Theorem.** 1) If $k_0$ is differentially closed, then there exists a unique field $K$, the PPV-extension, such that

- $K = k < z_{1,1}, \ldots, z_{n,n} >_{\Delta}$ for $Z = (z_{ij}) \in \text{GL}_n(K)$ with $\partial_0 Z = AZ$,
- $K_0 = k_0$.

2) The PPV-group $\text{Gal}_\Delta(K/k)$ of differential $k$—automorphisms of $K$ is a $\Delta'$-linear differential algebraic subgroup of $\text{GL}_n(k_0)$.

3) There is a Galois correspondence between $\Delta$ — differential subfields of $K$ containing $k$ and Kolchin $\Delta'$—closed subgroups of $\text{Gal}_\Delta(K/k)$.

4) Solvable by param. Liouvillian $\iff \text{Gal}_\Delta(K/k)$ solvable-by-finite.

5) For parameterized equations with regular singular points, the parameterized monodromy is Kolchin dense in the Galois group.

6) $\Delta.\text{tr.deg.}_kK = \Delta'$-diff.dim$_{k_0}\text{Gal}_\Delta(K/k)$. 
Ex. 5: Incomplete Gamma Function

\[ \gamma = \int_0^x s^{t-1} e^{-s} ds \] satisfies

\[ \frac{\partial^2 \gamma}{\partial x^2} - \frac{t-1-x}{x} \frac{\partial \gamma}{\partial x} = 0. \]

\[ k = \mathbb{C}(x, t), \quad \partial_0 = \frac{\partial}{\partial x}, \partial_1 = \frac{\partial}{\partial t}, \quad K = k(\gamma, \partial_1 \gamma, \partial_1^2 \gamma, \ldots) \]

\[ \text{Gal}_\Delta = \left\{ \left( \begin{array}{cc} 1 & a \\ 0 & b \end{array} \right) \mid a \in k_0, \ b \in k_0^*, \partial_1(\frac{\partial_1 b}{b}) = 0 \right\} \]

\[ = G_a(k_0) \times G_{\partial_1}^m \]

where \( G_{\partial_1}^m = \{ b \in k_0^* \mid \partial_1(\frac{\partial_1 b}{b}) = 0 \}. \)

\[ \{\partial_1\}\text{-diff.dim.}\text{Gal}_\Delta(K/k) = 1 \implies \gamma, \partial_1 \gamma, \ldots \text{ alg. ind. over } \mathbb{C}(x,t). \]

PPV-theory also holds for parameterized systems of integrable partial differential systems.
$k$ is a differential field of characteristic zero with derivations $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$

$k_0 = \{c \in k \mid \partial_0 c = 0\}$, differentially closed $\Delta' = \{\partial_1, \ldots, \partial_m\}$ field of $\partial_0$--constants

$C = \{c \in k \mid \partial_i c = 0 \text{ for all } i, 0 \leq i \leq m\}$
Integrable and Isomonodromic Families

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When is a PPV-group of the form \(G(C')\) for some linear differential algebraic group \(G\)?
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Proposition: Let $K$ be the PPV-extension of $k$ for the equation

$$\partial_0 Y = A_0 Y, \quad A_0 \in \text{M}_n(k).$$

The PPV-group $\text{Gal}_\Delta(K/k)$ is $G(C')$ for some linear differential algebraic group $G$ if and only if there exist $A_1, \ldots, A_m \in \text{M}_n(k)$ such that

$$\partial_i Y = A_i Y \quad i = 0, 1, \ldots, m$$

is an integrable system (i.e., $\partial_i A_j - \partial_j A_i = [A_j, A_i]$). In this case, the field $K$ is a usual $PV$—extension and the $G(C')$ is the PV-group.
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For $k = k_0(x)$ and regular singular points: integrable $\leftrightarrow$ isomonodromic
$2 \times 2$ Parameterized Families

$k_0 = \text{differential closed extension of } \mathbb{C}(t), \frac{\partial}{\partial t}$

$k = k_0(x), \text{derivations } \{\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\}.$
2 × 2 Parameterized Families

\( k_0 \) = differential closed extension of \( \mathbb{C}(t), \frac{\partial}{\partial t} \)

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With \( m > 1 \) parameters → Case 3: \( \exists \) new parameters \( \tilde{t}_1, \ldots, \tilde{t}_m \) and \( A_1, \ldots, A_s \in \text{M}_n(k), s \leq m \) such that

\[
\frac{\partial Y}{\partial x} = AY, \quad \frac{\partial Y}{\partial \tilde{t}_i} = A_iY, \quad i = 1, \ldots, s
\]

is an integrable system. If \( s = m, \frac{\partial Y}{\partial x} = AY \) is integrable.
Comments

- Peter Landesman - parameterized strongly normal extensions à la Kolchin.

- Alexey Ovchinnikov, Moshe Kamensky - Tannakian approach

- Anand Pillay - strongly normal extensions of ordinary differential fields with respect to definable sets other than constants. Generalized by Sanchez to partial differential fields.

- Bernard Malgrange, Hiroshi Umemura - general Galois theories.