Finite Time Emergence of a Shock Wave for Scalar Conservation Laws

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Abstract

For a convex conservation law

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \quad -\infty < x < \infty,\ t > 0,$$

bounded initial data $u_0(x)$, are considered that take on constant values u_- to the left of a bounded interval, and u_+ to the right, with $u_- > u_+$. The solution of the initial value problem is shown to collapse in finite time to a single shock wave joining u_- to u_+. The proof involves comparison with a solution having piecewise constant initial data for which the time of collapse is be calculated explicitly. This result has a significant application to steady granular flow in a chute, and the result is reformulated to apply to the Lighthill-Whitham-Richards equation of traffic flow.

1 Introduction

Consider the scalar conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

in which the flux $f : \mathbb{R} \to \mathbb{R}$ is smooth and convex: $f''(u) > 0$. In this paper, we demonstrate the emergence in finite time of a single shock wave, for L^∞ initial data $u_0(x)$, when the initial function u_0 is a constant u_- for $x < x_-$, and a constant u_+ for $x > x_+$, with $u_- > u_+,\ x_- < x_+$. That is, the weak solution converges in finite time to the piecewise constant solution consisting of a single shock wave from u_- to u_+. The proof involves the application of a comparison principle using generalized characteristics [1], a slightly different approach from that used in an earlier paper to prove a similar result [4]. An equivalent argument using the Lax formula [8] for weak solutions was used in [9] in the special case of stationary shock solutions of the inviscid Burgers equation.

Generalized characteristics were introduced by Dafermos in [1] as part of the analysis of the structure of solutions of scalar conservation laws; they were later used for more general balance laws and systems of equations [2, 3]. Generalized characteristics have proved useful in the theory of conservation laws as a means to study a variety of properties, including regularity of solutions and large-time behavior, often involving intricate and deep analysis of hyperbolic PDE. In this paper, we present a context in which properties of generalized characteristics are used in a simple way to prove a special result.

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In Section 2, we state and prove the main result. In Section 3, we discuss the application to size segregation in a steady shear flow of granular materials, the same application considered in [9], in which the finite time property has practical significance. In this section, we also illustrate the result in the context of the Lighthill-Whitham-Richards traffic flow model.

2 The Main Theorem

Suppose $u(x,t)$ is a weak solution of equation (1.1). Generalized characteristics are solutions (in the sense of Fillipov [5]) of the ODE

$$\frac{dx}{dt} = f'(u(x,t)). \tag{2.1}$$

As demonstrated in [1], generalized characteristics are either classical characteristics (i.e., C^1 solutions of (2.1)), or shocks. Through every point $(\overline{x},\overline{t})$ there is a unique forward characteristic (with $t > \overline{t}$), but backwards characteristics through $(\overline{x},\overline{t})$ need not be unique, and can form a wedge bounded by classical characteristics in the (x,t)-plane. The fastest of the backwards characteristics is called the *minimal characteristic*, and the slowest is called the *maximal characteristic*. Both minimal and maximal characteristics are classical, and are therefore straight lines on which $u(x,t)$ is constant. Moreover, for an initial value problem they extend backwards to $t = 0$.

Consider the initial value problem for (1.1), with initial data u_0 :

$$u_t + f(u)_x = 0, \quad -\infty < x < \infty, \;\; t > 0, \qquad u(x,0) = u_0(x), \quad -\infty < x < \infty. \tag{2.2}$$

We assume throughout that the flux function $f : \mathbb{R} \to \mathbb{R}$ is C^2 and convex: $f''(u) > 0$ for all $u \in \mathbb{R}$, and the initial function $u_0 : \mathbb{R} \to \mathbb{R}$ is in L^∞. We are concerned here with initial data that are constant outside some interval, as expressed in the following condition:
(H): There are constants $u_- > u_+,\ x_- < x_+$ such that $u_0(x) = u_-$ for $x < x_-$, and $u_0(x) = u_+$ for $x > x_+$.

In this section we prove that the solution of the initial value problem (2.2) converges to a piecewise constant shock

$$u(x,t) = \begin{cases} u_-, & x < \tilde{x} + s(t - \tilde{t}) \\ u_+, & x > \tilde{x} + s(t - \tilde{t}). \end{cases} \tag{2.3}$$

for $t \geq \tilde{t}$, in which $s = (f(u_+) - f(u_-))/(u_+ - u_-)$ is the shock speed and $\tilde{x}, \tilde{t}$ are to be determined.

To prove this result, we compare the solution $u(x,t)$ to the solution $\overline{u}(x,t)$ of a special initial value problem, for which the initial condition is

$$\overline{u}(x,0) = \overline{u}_0(x) = \begin{cases} u_-, & -\infty < x \leq x_- \\ u_m, & x_- < x \leq x_c \\ u_M, & x_c < x \leq x_+ \\ u_+, & x_+ < x < \infty. \end{cases} \tag{2.4}$$

Here, u_m, u_M, x_c are parameters satisfying

$$u_m \leq u_+ < u_- \leq u_M, \qquad x_- \leq x_c \leq x_+. \tag{2.5}$$

They will be specified later in terms of u_0.

The solution $\overline{u}(x,t)$ of the special initial value problem is elementary. Shocks S_-, S_+ propagate from $(x_-,0), (x_+,0)$ in the (x,t) plane, and a rarefaction wave is centered at $(x_c,0)$ Upon interacting with the rarefaction wave, the shock S_- accelerates, whereas the shock S_+ decelerates. The two shocks collide at a point $(\tilde{x},\tilde{t})$, from which the single shock S, given by (2.3), emerges. This structure is illustrated in Fig. 1.

The minimal and maximal backwards characteristics originating at $(\tilde{x},\tilde{t})$ intersect the x-axis at points $(y_-,0), (y_+,0)$, respectively. Moreover, there is a straight line characteristic embedded in the

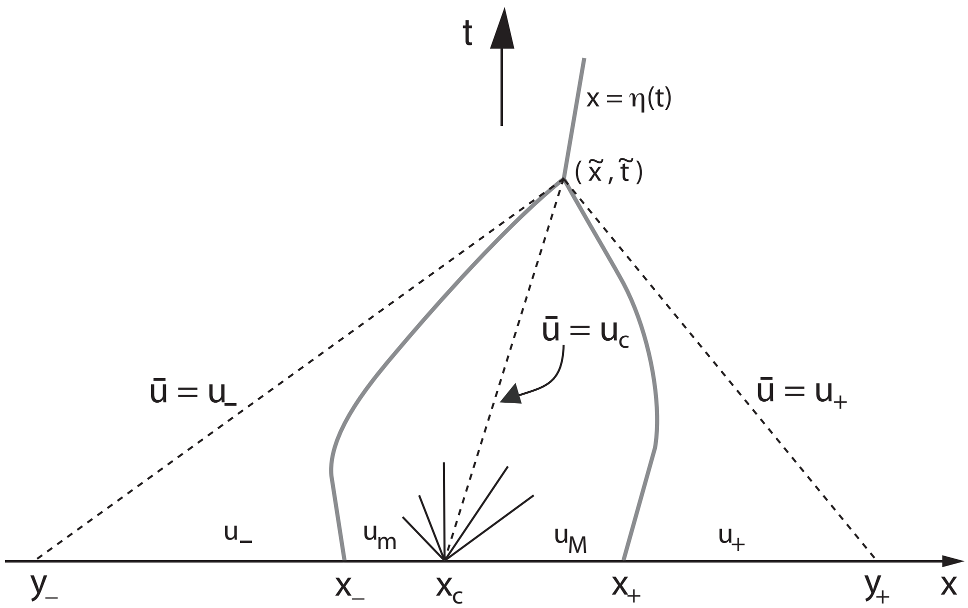

Figure 1: The comparison solution.

centered rarefaction, joining $(\tilde{x}, \tilde{t})$ to $(x_c, 0)$. On this characteristic, $\overline{u}(x,t)$ is a constant u_c, to be determined along with $\tilde{x}, \tilde{t}, y_\pm$. Since the three characteristics entering $(\tilde{x}, \tilde{t})$, are all straight lines with speed $f'(\overline{u}(x,t))$, on which $\overline{u}(x,t)$ is a constant, we obtain three expressions for $\tilde{x}$:

$$\tilde{x} = y_- + \tilde{t} f'(u_-) = x_c + \tilde{t} f'(u_c) = y_+ + \tilde{t} f'(u_+). \tag{2.6}$$

These three equations are supplemented by a pair of equations obtained by integrating the PDE (1.1) over the triangles with vertices $(\tilde{x}, \tilde{t}), (x_c, 0), (y_-, 0)$ and $(\tilde{x}, \tilde{t}), (x_c, 0), (y_+, 0)$, using the divergence theorem in (x,t) to express the area integrals (which are zero) as sums of integrals along the three edges of the two triangles:

$$\tilde{t}\,[f(u_c) - u_c f'(u_c) - f(u_-) + u_- f'(u_-)] \;=\; (x_- - y_-)u_- + (x_c - x_-)u_m, \tag{2.7a}$$

$$\tilde{t}\,[f(u_+) - u_+ f'(u_+) - f(u_c) + u_c f'(u_c)] \;=\; (y_+ - x_+)u_+ + (x_+ - x_c)u_M. \tag{2.7b}$$

Lemma 2.1. *For each choice of parameters $u_\pm, x_\pm, x_c, u_m, u_M$ satisfying (2.5), there is a unique solution $(\tilde{x}, \tilde{t}, y_-, y_+, u_c)$ of equations (2.6),(2.7) satisfying $u_+ \le u_c \le u_-, y_- < x_- < x_+ < y_+$.*

Proof: From (2.6), we obtain expressions for $y_\pm$:

$$y_- = x_c + \tilde{t}[f'(u_c) - f'(u_-)], \quad y_+ = x_c + \tilde{t}[f'(u_c) - f'(u_+)]. \tag{2.8}$$

Substituting into the equations (2.7) yields

$$\tilde{t}[f(u_c) - f(u_-) - f'(u_c)(u_c - u_-)] \;=\; (u_m - u_-)(x_c - x_-) \stackrel{\text{def}}{=} -\gamma^2 < 0 \tag{2.9a}$$

$$\tilde{t}[f(u_+) - f(u_c) - f'(u_c)(u_+ - u_c)] \;=\; (u_M - u_+)(x_+ - x_c) \stackrel{\text{def}}{=} \delta^2 > 0 \tag{2.9b}$$

Note that γ^2, δ^2 depend only on the given parameters. Eliminating $\tilde{t}$, we reduce to a single equation for u_c :

$$G(u_c) \equiv [f(u_+) - f(u_c) - f'(u_c)(u_+ - u_c)]\gamma^2 + [f(u_c) - f(u_-) - f'(u_c)(u_c - u_-)]\delta^2 = 0. \tag{2.10}$$

Using convexity of f, we easily find $G(u_+) < 0$, and $G(u_-) > 0$, establishing the existence of a solution $u_c \in (u_-, u_+)$. Uniqueness follows from the calculation

$$G'(u_c) = \left[\gamma^2(u_c - u_+) + \delta^2(u_- - u_c)\right] f''(u_c) > 0.$$

The remaining unknowns are given by (2.6) (for $\tilde{x}$), (2.8), and (2.9) (for $\tilde{t}$). ■

We can illustrate the Lemma for Burgers' equation

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0 \tag{2.11}$$

by calculating the unknown parameters explicitly. Here, $f(u) = \frac{1}{2}u^2$, so that equations (2.9) take the form

$$\tilde{t}(u_- - u_c)^2 = 2(x_c - x_-)(u_- - u_m) \stackrel{\text{def}}{=} \gamma^2 \tag{2.12a}$$

$$\tilde{t}(u_+ - u_c)^2 = 2(x_+ - x_c)(u_M - u_+) \stackrel{\text{def}}{=} \delta^2 \tag{2.12b}$$

Then

$$u_c = \frac{\gamma u_+ + \delta u_-}{\gamma + \delta}, \quad \tilde{t} = \frac{(\gamma+\delta)^2}{(u_- - u_+)^2}, \quad \tilde{x} = x_c + \frac{(\gamma+\delta)(\gamma u_+ + \delta u_-)}{(u_- - u_+)^2}. \tag{2.13}$$

Returning to the general case, let $u_0(x)$ be the initial data satisfying (H). We compare the solution $u(x,t)$ to the solution $\overline{u}(x,t)$ with initial data $\overline{u}_0(x)$ given by (2.4), where u_m, u_M are chosen with

$$u_m \le \operatorname*{ess-inf}_{-\infty<x<\infty} u_0(x), \quad u_M \ge \operatorname*{ess-sup}_{-\infty<x<\infty} u_0(x), \tag{2.14}$$

and x_c is chosen specifically so that

$$\int_{x_-}^{x_+} \overline{u}_0(x)\,dx = (x_c - x_-)u_m + (x_+ - x_c)u_M = \int_{x_-}^{x_+} u_0(x)\,dx. \tag{2.15}$$

We note that for any $\overline{y}$ outside the interval (x_-, x_+),

$$\int_{\overline{y}}^{y} [u_0(x) - \overline{u}_0(x)]\,dx \ge 0, \qquad -\infty < y < \infty. \tag{2.16}$$

This property is obvious from (2.14), (2.15) except perhaps for $x_c < y < x_+$. In this case,

$$0 = \int_{\overline{y}}^{x_+} [u_0(x) - \overline{u}_0(x)]\,dx = \int_{\overline{y}}^{y} [u_0(x) - \overline{u}_0(x)]\,dx + \int_{y}^{x_+} [u_0(x) - \overline{u}_0(x)]\,dx.$$

Thus,

$$\int_{\overline{y}}^{y} [u_0(x) - \overline{u}_0(x)]\,dx = -\int_{y}^{x_+} [u_0(x) - \overline{u}_0(x)]\,dx = \int_{y}^{x_+} [u_M - u_0(x)]\,dx \ge 0.$$

We can now prove the main result:

Theorem 2.2. *Under assumption (H) on the initial data, there is a value $\tilde{x}$ and a time $\tilde{t} < \infty$ such that for $t \ge \tilde{t}$, $u(x,t)$ is given by (2.3).*

Proof: Let $\tilde{x}, \tilde{t}$ be given by Lemma 2.1, where u_m, u_M are specified, satisfying (2.14). Let $\eta(t) = \tilde{x} + s(t - \tilde{t})$, and fix $(\hat{x}, \hat{t})$ with $\hat{t} \ge \tilde{t}, -\infty < \hat{x} < \eta(\hat{t})$, so that $(\hat{x}, \hat{t})$ is a point of continuity of $\overline{u}(x,t)$. The unique backward characteristic of $\overline{u}$ originating at $(\hat{x}, \hat{t})$ is a straight line with speed $f'(u_-)$, intersecting the $x-$axis at a point $\overline{y} < y_-$. (See Fig. 2.) We claim that $u(\hat{x}, \hat{t}) = u_-$. Suppose for a contradiction that $u(\hat{x}, \hat{t}) = \hat{u} \ne u_-$. Then the unique backward characteristic of u originating at

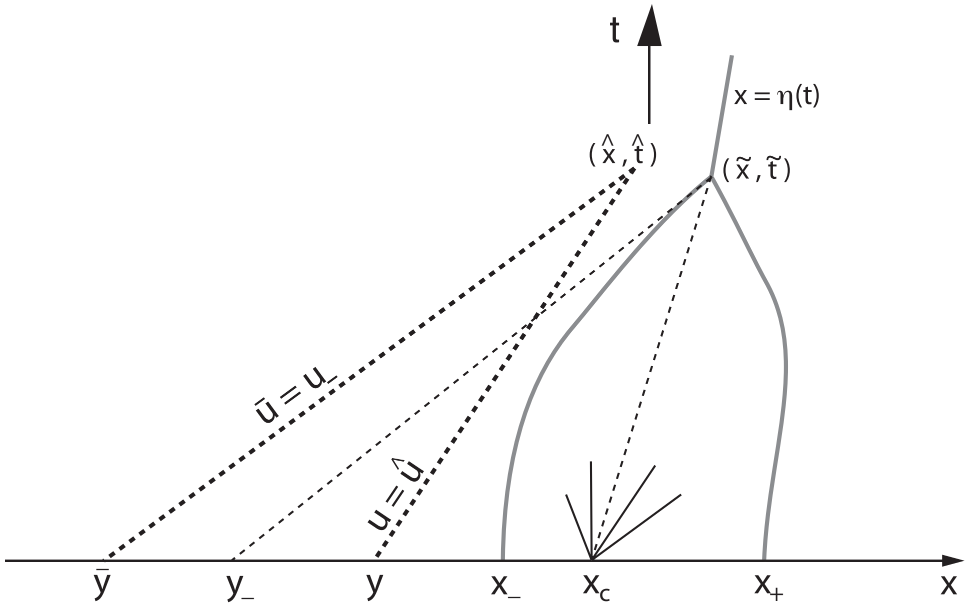

Figure 2: The comparison of two solutions.

$(\hat{x}, \hat{t})$ would be a straight line with speed $f'(\hat{u})$, intersecting the $x-$axis at a point $y \neq \overline{y}$ ($y < \overline{y}$ if $\hat{u} > u_-$, or $y > \overline{y}$ if $\hat{u} < u_-$). Integrating the equation

$$(u - \overline{u})_t + [f(u) - f(\overline{u})]_x = 0 \tag{2.17}$$

over the triangle with vertices $(\hat{x}, \hat{t}), (\overline{y}, 0), (y, 0)$ (see Fig. 2), and applying the divergence theorem, we get the equation

$$\int_0^{\hat{t}} \{f(\hat{u}) - f(\overline{u}) - f'(\hat{u})(\hat{u} - \overline{u})\}\, dt + \int_0^{\hat{t}} \{f(u_-) - f(u) - f'(u_-)(u_- - u)\}\, dt = \int_{\overline{y}}^{y} (u_0(x) - \overline{u}_0(x))\, dx. \tag{2.18}$$

(The equation is valid in both cases $y < \overline{y}$ and $y > \overline{y}$.) By convexity of f, the left hand side is strictly negative. But this contradicts (2.16), thus establishing the claim that $u(\hat{x}, \hat{t}) = u_-$. Similarly, for $\hat{t} \geq \tilde{t}, \eta(\hat{t}) < \hat{x} < \infty$, we can show $u(\hat{x}, \hat{t}) = u_+$. ■

Remark. If the far field conditions $u_\pm$ are reversed: i.e., if $u_- < u_+$, then the solution converges to a rarefaction wave joining u_- to u_+ as $t \to \infty$ [11] . However, the solution in general need not be continuous at any time, as the following example illustrates. Let

$$u_0(x) = \begin{cases} -1, & \text{if } \; x < -1, \; \text{ or } \; 0 < x < 1 \\ \\ 1, & \text{if } \; -1 < x < 0, \; \text{ or } \; x > 1 \end{cases} \tag{2.19}$$

be initial data for Burgers' equation (2.11). The solution of the initial value problem (2.11), (2.19) consists of a pair of rarefaction waves that gradually reduce the strength of a stationary shock located at $x = 0$. For $t > 1$, the solution is

$$u(x,t) = \begin{cases} -1, & x < -(t+1) \\ (x+1)/t, & -(t+1) < x < 0 \\ (x-1)/t, & 0 < x < t+1 \\ 1, & x > t+1. \end{cases}$$

In this example, the shock has strength $2/t > 0$ for all $t > 1$, so that the single rarefaction wave is not established in finite time.

3 Application to Granular Flow and Traffic Flow

Granular flow. In this application, we consider the segregation of a mixture of small and large particles by a process of *kinetic sieving* in an avalanche. In a rock or debris avalanche down a hillside, or the flow of granular material on a slope, such as in processing powders, or tablet pharmaceuticals, the speed in the avalanche (parallel to the hillside) varies roughly linearly with depth. Assuming a known velocity profile $v(z)$, an equation expressing the evolution of the volume fraction of small particles was formulated by Gray and Thornton [6]:

$$\partial_t\phi + v(z)\partial_x\phi + S\partial_z[\phi(\phi-1)] = 0, \quad x > 0,\ 0 < z < L. \tag{3.1}$$

We also assume the depth L of the flow (the height of the free surface) is constant, and that $v(z)$ is a monotonically increasing smooth function of height z above the solid surface $z = 0$. Since no particles penetrate the upper and lower boundaries, we assume no flux boundary conditions, which we take in the form:

$$\phi(x,0) = 1; \quad \phi(x,L) = 0. \tag{3.2}$$

Let's consider steady flow, in which $\phi = \phi(x,z)$ is independent of t, and satisfies an inlet boundary condition

$$\phi(0,z) = \phi_0(z), \quad 0 < z < L, \tag{3.3}$$

in which $\phi_0 : [0,L] \to [0,1]$ is a given L^∞ function. This boundary condition could be provided for example by a feeder or hopper at the head of a chute, with $x > 0$ being the distance down the chute.

Let $\psi(z) = \int_0^z v(\eta)\,d\eta$, and define new variables $t = Sx/\psi(L), y = 2\psi(z)/\psi(L) - 1, u = 2\phi - 1$. Then (3.1) is transformed into Burgers equation for $u = u(y,t)$:

$$\partial_t u + \partial_y(\tfrac{1}{2}u^2) = 0, \quad -1 < y < 1,\ t > 0, \tag{3.4}$$

with boundary conditions

$$u(-1,t) = 1,\ u(1,t) = -1. \tag{3.5}$$

The inlet boundary condition now becomes an initial condition for Burgers equation:

$$u(y,0) = u_0(y), \quad -1 < y < 1, \tag{3.6}$$

where $u_0(y) = 2\phi_0(z) - 1$.

To apply Theorem 2.2, we extend the initial data to be constant outside the interval $-1 < y < 1$: let $u_0(y) = 1,\ y < -1$, and $u_0(y) = -1,\ y > 1$. We can calculate $\tilde{t}$ as a function of y_c, given by $y_c = \int_{-1}^{1}$. The solution $u = \overline{u}(y,t)$, including the curved shocks, can be found explicitly. The rarefaction wave is $\overline{u}(y,t) = (y - y_c)/t$, so that the times $t_\pm$ of intersection of the leading and trailing characteristics, on which $\overline{u} = \pm 1$, with the shocks at $y = \pm 1$ are given by $t_\pm = 1 \mp y_c$. The curved shocks $y = y_\pm(t), t > t_\pm$, are given by ODEs expressing the Rankine Hugoniot condition:

$$y'_\pm(t) = \tfrac{1}{2}(\mp 1 + (y - y_c)/t), \quad x_\pm(t_\pm) = \pm 1,$$

with solutions

$$y_\pm(t) = x_c \mp t \pm 2\sqrt{(1 \mp y_c)t}.$$

The shocks meet at $y = \tilde{y} = -y_c$, when $t = \tilde{t} = 2 + 2\sqrt{1 - y_c^2}$. Then $\tilde{t}(y_c) \le 4$, with equality only for $y_c = 0$.

Correspondingly, the mixture entering the chute at $x = 0$ has fully segregated before the particles have traveled a distance $x = 4\psi(L)/S$. Conservation of mass gives the height z^* of the interface separating the layer of small grains that settles below the layer of large grains: $\psi(z^*) = \int_0^L v(z)\phi_0(z)\,dz$. This application was also worked out in [9] using the Lax formula [7].

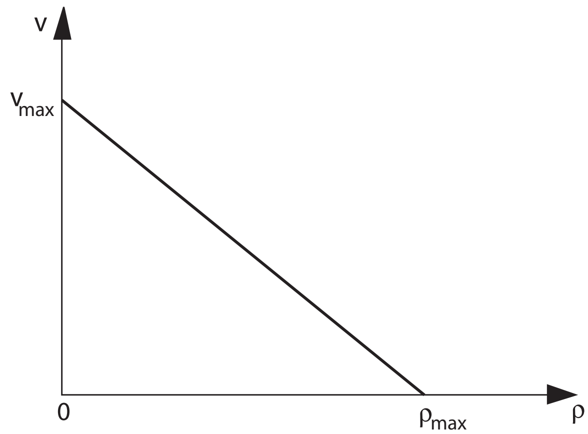

Figure 3: Velocity v as a decreasing linear function of car density ρ in traffic flow.

Traffic flow. The Lighthill-Whitham-Richards (LWR) model of traffic flow [10] provides a second illustration of Theorem 2.2. Let $\rho(y,t)$ denote the density of traffic on a highway (i.e., the number of cars per unit distance) at location y and time t. Assuming the velocity $v(\rho)$ depends on the density of traffic, the LWR model is

$$\partial_\tau \rho + \partial_y(\rho v(\rho)) = 0. \tag{3.7}$$

We will assume that v is a decreasing smooth function of ρ in an interval $0 \le \rho \le \rho_{max}$, where ρ_{max} is the maximum density, for example in a stationary traffic jam: $v(\rho_{max}) = 0$.

Now suppose that at an initial time, say $\tau = 0$, there are two locations $y_b < y_a$ on the highway such that the traffic density ρ_b behind (i.e., $y < y_b$) is constant, and smaller than the constant traffic density ρ_a ahead (i.e., $y > y_a$), and that the density is everywhere bounded between two values: $\rho_m \le \rho_0 \le \rho_M$. Then, allowing for the fact that the nonlinearity is concave rather than convex, the theorem guarantees that after some time $\tilde{\tau}$, the traffic flow will consist of a single shock wave $y = s\tau + \tilde{y}_0$ with $\rho = \rho_b$ behind the wave and $\rho = \rho_a$ ahead.

When $v(\rho)$ is linear (see Fig. 3), the LWR equation has a quadratic nonlinearity, and can be transformed into Burgers equation using scaling and translation of the variables. Specifically, the change of variables

$$u = \rho - \tfrac{1}{2}\rho_{max};\ x = -y;\ t = \frac{2v_{max}}{\rho_{max}}\tau$$

yields Burgers equation (2.11). The corresponding constants in §2 are:

$$u_- = \rho_a - \tfrac{1}{2}\rho_{max};\ u_+ = \rho_b - \tfrac{1}{2}\rho_{max};\ u_m = \rho_m - \tfrac{1}{2}\rho_{max};\ u_M = \rho_M - \tfrac{1}{2}\rho_{max};\ x_- = -y_b;\ x_+ = -y_a.$$

Suppose the initial traffic density ρ_0 is bounded by the density ρ_a ahead and the density ρ_b behind, so that $\rho_m = \rho_b, \rho_M = \rho_a$. Let $y_c = -x_c$ be defined by the initial mass $\int_{y_b}^{y_a} \rho_0(y)\,dy$:

$$y_c = \frac{1}{\rho_a - \rho_b}\left\{ y_b\rho_a - y_a\rho_b - \int_{y_b}^{y_a} \rho_0(y)\,dy \right\}.$$

Then formulae (2.13) yield

$$\tilde{\tau} = \frac{\rho_{max}}{v_{max}(\rho_a - \rho_b)}\left[y_a - y_b + 2\sqrt{(y_a - y_c)(y_c - y_b)} \right].$$

The maximum estimate (the largest $\tilde{\tau}$) is achieved when $y_c = \frac{1}{2}(y_a + y_b)$, so that

$$\tilde{\tau} = \frac{2\rho_{max}(y_a - y_b)}{v_{max}(\rho_a - \rho_b)}.$$

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