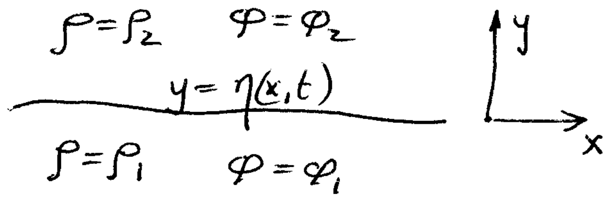
Acheson 3.2, 3.6

3.2

Linearized Eqns.

$$\Delta \varphi_i = 0 \qquad (1)$$

$$\frac{\partial \varphi_i}{\partial y} = \frac{\partial \eta}{\partial t}, \quad y = 0 \qquad (2)$$



pressure $p_1 = p_2$ on $y = \eta$:

$$\rho_1 \frac{\partial \varphi_1}{\partial t} + \rho_1 g \eta = \rho_2 \frac{\partial \varphi_2}{\partial t} + \rho_2 g \eta \qquad \text{on } y = 0 \quad (3)$$

Seek solutions of (1) $\varphi_i(x, y, t) = f_i(y) \sin(kx - \omega t)$,

so $\quad f_i'' - k^2 f_i = 0$

$$\therefore \quad f_1(y) = A e^{ky}, \quad \text{bounded as } y \to -\infty$$

$$f_2(y) = B e^{-ky} \quad \text{"} \quad y \to +\infty$$

Then $(u, v) = \underset{\sim}{\nabla} \varphi \to (0,0)$ as $y \to \pm\infty$.

Let $\quad \eta = C \cos(kx - \omega t)$.

Then

$$\left. \begin{aligned} kA &= +\omega C \\ -kB &= \omega C \end{aligned} \right\} \quad (2)$$

$$\therefore \quad A = -B$$

Eqn. (3): $\quad -\rho_1 \omega A + \rho_1 g C = -\rho_2 B \omega + \rho_2 g C$

$$\therefore \quad (\rho_1 - \rho_2) g C = (\rho_2 + \rho_1) \omega A \qquad (4)$$

$$\therefore \quad \begin{vmatrix} k & -\omega \\ (\rho_1 + \rho_2)\omega & (\rho_2 - \rho_1) g \end{vmatrix} = 0$$

3.2 (Cont'd)

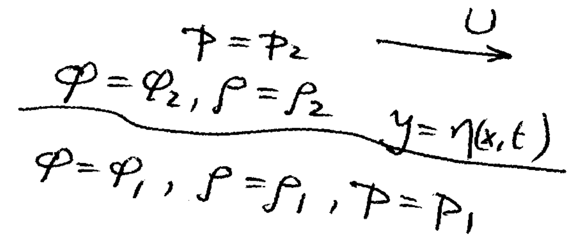
I.e., $$(\rho_2-\rho_1)kg + (\rho_1+\rho_2)\omega^2 = 0$$

$$\omega^2 = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}kg$$

$$\therefore \quad c^2 = \frac{\omega^2}{k^2} = \frac{g}{k}\frac{\rho_1-\rho_2}{\rho_1+\rho_2}$$

3.6 Kelvin–Helmholtz instability

Surface tension T:



$$p_1 = p_2 - T\frac{\partial^2\eta}{\partial x^2} \qquad (1)$$

Incorporate wind speed U: replace velocity potential φ_2 by $\varphi_2 + Ux$, so that $\underset{\sim}{\nabla}\varphi_2$ is a small perturbation to the uniform velocity $(U, 0)$.

Then $$\Delta\varphi_i = 0, \quad i = 1,2 \qquad (2)$$

interface conditions: $y = \eta(x,t)$, so $\dot{y} = \eta_x\dot{x} + \eta_t$.

Below: $$\frac{\partial\varphi_1}{\partial y} = \frac{\partial\eta}{\partial t} \quad \text{(linearizing)} \qquad (3), \quad y = 0$$

Above: $$\frac{\partial\varphi_2}{\partial y} = \eta_x\left(U + \frac{\partial\varphi_2}{\partial x}\right) + \eta_t \simeq U\eta_x + \eta_t \quad \text{(linearizing)} \qquad (4), \quad y = 0.$$

Bernoulli condition This follows from

$$\frac{\partial\varphi_i}{\partial t} + \frac{1}{2}|\underset{\sim}{u}_i|^2 + \frac{p_i}{\rho_i} + g\eta = 0 \qquad i = 1,2 \qquad (5)$$

We use (1), and the linearization of (5),

in which

$$\frac{1}{2}\left|\underset{\sim}{u}_2\right|^2 \sim \frac{1}{2}U^2 + U\frac{\partial \varphi_2}{\partial x},$$

$\frac{1}{2}U^2$ can be incorporated into $\varphi_2 \longrightarrow \varphi_2 + \frac{1}{2}U^2 t$.

Then eqn.(1) becomes

$$\rho_1 \frac{\partial \varphi_1}{\partial t} + \rho_1 g \eta = \rho_2 \frac{\partial \varphi_2}{\partial t} + \rho_2 U \frac{\partial \varphi_2}{\partial x} + \rho_2 g \eta + T\eta_{xx} \qquad (6).$$

Now seek complex solutions:

$$\left.\begin{array}{ll} y<0: & \varphi_1 = A e^{ky} e^{i(kx-\omega t)} \\ y>0: & \varphi_2 = B e^{-ky} e^{i(kx-\omega t)} \end{array}\right\} \text{ solve (2) + are bounded,}$$

$$\eta = C e^{i(kx-\omega t)}$$

Then

$$\left.\begin{array}{ll} (3): & kA = -i\omega C \\ (4): & -kB = ikCU - i\omega C \end{array}\right\} (7)$$

(6): $\quad -\rho_1 i\omega A + \rho_1 g C = -\rho_2 \omega i B + \rho_2 U k i B + \rho_2 g C - k^2 T C$

Substitute (7) into (6): $\quad iA = \frac{\omega}{k}C, \quad iB = \left(U - \frac{\omega}{k}\right)C$:

$$-\rho_1 \frac{\omega^2}{k} + \rho_1 g = -\rho_2 \omega\left(U - \frac{\omega}{k}\right) + \rho_2 U k\left(U - \frac{\omega}{k}\right) + \rho_2 g - k^2 T$$

Multiply by k and collect terms:

$$(\rho_1+\rho_2)\omega^2 - 2\rho_2 k U \omega - (\rho_1-\rho_2) g k + \rho_2 U^2 k^2 - k^3 T = 0. \tag{8}$$

This is a quadratic equation for $\omega = \omega(k)$. If ω is not real, then $e^{-i\omega t}$ will grow exponentially for one of the roots $\omega = \omega_R \pm i\omega_I$. Therefore, the interface is unstable if $\omega_I(k) \neq 0$ for any k.

The discriminant of (8) is

$$\begin{aligned} D &= \rho_2^2 k^2 U^2 - (\rho_1+\rho_2)\{\rho_2 U^2 k^2 - (\rho_1-\rho_2) g k - k^3 T\} \\ &= k\left[(\rho_1+\rho_2) k^2 T - \rho_1\rho_2 U^2 k + (\rho_1^2-\rho_2^2) g\right] = k\,Q(k), \end{aligned}$$

where $Q(k)$ is the quadratic expression.

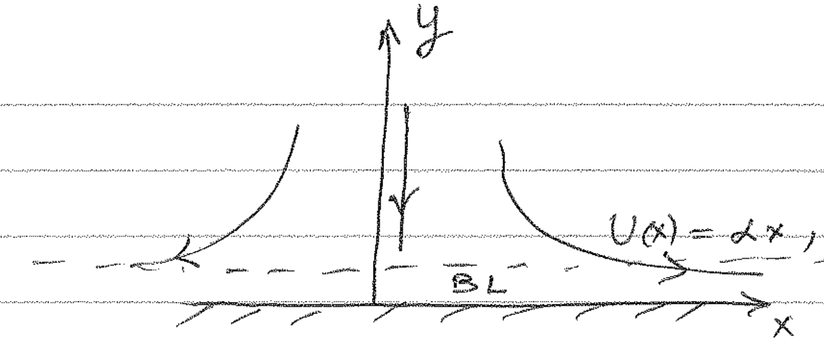
<u>Instability</u> appears for a range of k if minimum of $Q(k)$ gives $D < 0$. Min $Q(k)$: $Q'(k) = 0$,

$$\text{i.e. } k^* = \frac{1}{2(\rho_1+\rho_2)T}\rho_1\rho_2 U^2$$

$$Q(k^*) = -\frac{1}{4}\frac{(\rho_1\rho_2 U^2)^2}{(\rho_1+\rho_2)T} + (\rho_1^2-\rho_2^2) g < 0$$

$$\text{if } U^2 > 2\,\frac{\rho_1+\rho_2}{\rho_1\rho_2}\sqrt{(\rho_1-\rho_2) g T}.$$

8.2



(1) $u u_x + v u_y = -\frac{1}{\rho} p_x + \nu u_{yy}$ $\qquad p = p(x)$.

(2) $u_x + v_y = 0$

Large y: $u = U(x) = \alpha x$, $v = 0$.

$\therefore \quad -\frac{1}{\rho} p_x = u u_x = \alpha^2 x$.

Let $u = \alpha x f'(\eta)$, $\eta = \frac{y}{g(x)}$. Then $f'(\infty) = 1$.

Then $u_x = \alpha f'(\eta) - \frac{\alpha x g'}{g^2} y f''(\eta) = -v_y$ from (2)

$\rightarrow \alpha$ for large y only if $g' = 0$: $g = \text{const.}$

$\therefore \quad v = -\alpha g f(\eta) + k(x)$.

$v = 0$ at $y = 0$ $\Rightarrow$ $k = \text{const.}$, so can be folded into f:

(3) $f(0) = 0$.

Eqn. (1) becomes, using $u_y = \frac{\alpha x}{g} f''$, $u_{yy} = \frac{\alpha x}{g^2} f''$

$$\alpha^2 x f'^2 - \alpha^2 x f f' = \alpha^2 x + \frac{\nu \alpha x}{g^2} f'''$$

$$\therefore \quad f'^2 - f f'' = 1 + \frac{\nu}{\alpha g^2} f''' = 1 + f''' \text{ if } g = \sqrt{\frac{\nu}{\alpha}}$$

Moreover, $u = 0$ on $y = 0$ $\Rightarrow$ $f'(0) = 0$.