

3-d. Write $\tilde{X} = (\xi, \eta, \zeta)$ "xi, eta, zeta"
 $\underline{x} = (x, y, z)$. (see Chorin & Marsden)

strain tensor = "displacement gradient" = $\frac{\partial \tilde{X}}{\partial \underline{x}}$

Jacobian $J = \det\left(\frac{\partial \tilde{X}}{\partial \underline{x}}\right) = \begin{vmatrix} \xi_x & \eta_x & \zeta_x \\ \xi_y & \eta_y & \zeta_y \\ \xi_z & \eta_z & \zeta_z \end{vmatrix}$

$$= \xi_x \eta_y \zeta_z + \xi_y \eta_z \zeta_x + \xi_z \eta_x \zeta_y \\ - \xi_x \eta_z \zeta_y - \xi_y \eta_x \zeta_z - \xi_z \eta_y \zeta_x$$

multilinear in columns i.e. in ξ, η, ζ .

LEMMA $\frac{\partial J}{\partial t}(x, t) = J(x, t) \operatorname{div}_{\tilde{X}} u(\tilde{X}(x, t), t)$

Proof $\frac{\partial}{\partial t} J = \begin{vmatrix} \partial_t \xi_x & \eta_x & \zeta_x \\ \partial_t \xi_y & \eta_y & \zeta_y \\ \partial_t \xi_z & \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \partial_t \eta_x & \zeta_x \\ \xi_y & \partial_t \eta_y & \zeta_y \\ \xi_z & \partial_t \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \xi_x & \eta_x & \partial_t \zeta_x \\ \xi_y & \eta_y & \partial_t \zeta_y \\ \xi_z & \eta_z & \partial_t \zeta_z \end{vmatrix}$

Now $\frac{\partial}{\partial t} \xi_x = \frac{\partial}{\partial x} \frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x} u \quad \left\{ \tilde{u} = (u, v, w) \right\}$

etc. $\frac{\partial}{\partial t} \eta_y = \frac{\partial}{\partial y} v \dots$

Then
$$\frac{\partial J}{\partial t} = \begin{vmatrix} u_x & \eta_x & \zeta_x \\ u_y & \eta_y & \zeta_y \\ u_z & \eta_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \zeta_x & v_x & \zeta_x \\ \zeta_y & v_y & \zeta_y \\ \zeta_z & v_z & \zeta_z \end{vmatrix} + \begin{vmatrix} \zeta_x & \eta_x & w_x \\ \zeta_y & \eta_y & w_y \\ \zeta_z & \eta_z & w_z \end{vmatrix}$$

But
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta} \frac{\partial \zeta}{\partial x} \quad (\text{chain rule})$$

since $u = u(\underline{X}, t) \quad \underline{X} = (\xi, \eta, \zeta) = \underline{X}(x, t)$

1st det: use linearity.

$$\begin{vmatrix} u_x & \eta_x & \zeta_x \\ u_y & \eta_y & \zeta_y \\ u_z & \eta_z & \zeta_z \end{vmatrix} = \begin{vmatrix} u_\xi \zeta_x & \eta_x & \zeta_x \\ u_\xi \zeta_y & \eta_y & \zeta_y \\ u_\xi \zeta_z & \eta_z & \zeta_z \end{vmatrix} + \underbrace{\begin{vmatrix} u_\eta \eta_x & \eta_x & \zeta_x \\ u_\eta \eta_y & \eta_y & \zeta_y \\ u_\eta \eta_z & \eta_z & \zeta_z \end{vmatrix}}_0 + \underbrace{\begin{vmatrix} u_\zeta \zeta_x & \eta_x & \zeta_x \\ u_\zeta \zeta_y & \eta_y & \zeta_y \\ u_\zeta \zeta_z & \eta_z & \zeta_z \end{vmatrix}}_0$$

$$= u_\xi J$$

sum:
$$\frac{\partial J}{\partial t} = u_\xi J + u_\eta J + u_\zeta J$$

$$\underline{\underline{= J \operatorname{div} \underline{u}. \quad \#}}$$