F(x) = x + x^3

F'(x) = 1 + 3x^2 
F''(x) = 6x 
F'''(x) = 6

F''(0) = 6 > 0 \Rightarrow x = 0 is repelling

F(x) = x - x^3

F'(x) = 1 - 3x^2 
F''(x) = -6x 
F'''(x) = -6

F''(0) = 0 \Rightarrow x = 0 is attracting

p.62 #1 (a) F_1(x) = x + x^2 + \lambda \quad \text{at } \lambda = 0

F_1'(x) = 1 + 2x \quad \frac{\partial F_1}{\partial \lambda} = 1

F_1''(x) = 2

Fixed point set: \{(x, \lambda); x = x + x^2 + \lambda \} = \{(x, \lambda); \lambda = -2x^2 \} looks like tangent saddle-node bifurcation at \((\lambda_0, x) = (0, 0)\).

Check:

\[ F_0'(x) = 0 \]

1) \[ F_{\lambda_0}'(0) = F_0'(0) = 1 \]
2) \[ F_{\lambda_0}''(0) = 2 \neq 0 \]
3) \[ \frac{\partial F_1}{\partial \lambda}(x_0, 0) = 1 \neq 0 \]

For \( x = \mu x \), \( p''(0) = -\frac{F_0''(0)}{1} = -2 \Rightarrow \text{saddle curve opens left} \)

p.50 #4 (f) \[ F_{\mu}(x) = \mu x + x^3 \quad \text{at } \mu = 1 \]

\[ F_{\mu}'(x) = \mu + 3x^2 \quad F_{\mu}''(x) = 6x \quad F_{\mu}'''(x) = 6 \]

Fixed point set: \{(x, \mu); \mu x + x^3 = x^3 \} = \{(x, \mu); \alpha (\mu - 1 + x^2) = 0 \} looks like pitchfork bifurcation at \((\mu_0, x) = (1, 0)\).

Check:

1) \[ G_{\mu_0}(0) = 0 \neq 0 \]
2) \[ \frac{G_{\mu_0}''(0) = 6 \neq 0 \Rightarrow \text{Pitchfork bifurcation at } \mu_0 = \frac{-G_{\mu_0}''(0)}{\frac{3}{\frac{\partial G_{\mu_0}'}{\partial \mu}}(0)} = \frac{-6}{3} = -2} \]
3) \[ \frac{G_{\mu_0}''(0) = 0}{\frac{\partial G_{\mu_0}''}{\partial \mu}} = 1 \neq 0 \]
Transcritical Bifurcation

Consider

(i) \( f'(\lambda(0)) = 0 \quad \forall \lambda \in \beta_0(\lambda_0) \)

(ii) \( f'(\lambda(0)) = 1 \)

(iii) \( f''(\lambda(0)) \neq 0 \)

(iv) \( \frac{d}{d\lambda} [f'(\lambda(0)) etc.]

Thus \( f'(\lambda_0) \neq 0 \) and

\( \exists (\lambda_0, \xi) \in \beta_0(\lambda_0, 0) : f'(\lambda_0) = \xi \) and \( \xi \neq 0 \)

\( \xi = \{ (\lambda, \xi) \in \beta_0 : \xi = f'(\lambda_0) \} \)

Ex.

\( f(x) = ax(1-x) \) at \( x=0 \) and \( a=1 \)

Show Trans. Bif. and draw local bif. curve.

Proof:

Define \( G(\lambda, \xi) = f'(\lambda, \xi) - \xi \) and remove \( \xi = 0 \).

Define \( H(\lambda, \xi, \eta) \)

\( H(\lambda, 0, 0) = \frac{d}{d\xi} [f'(\lambda_0) - \eta] \)

\( \lim_{\xi \to 0} G(\lambda_0, 0) = \lim_{\eta \to 0} H(\lambda_0, 0, \eta) \)

Show \( 0 = H(\lambda, \xi, \eta) \) can be solved for \( \xi \) as function of \( \lambda \).

Need \( \frac{dH}{d\xi} \neq 0 \). But \( \frac{dH}{d\xi} (\lambda_0, 0) = \frac{d}{d\xi} G(\lambda_0, 0, 0) \)

\( \frac{d}{d\xi} G(\lambda_0, 0) = \frac{d}{d\xi} [f'(\lambda_0) - \xi] \)

\( \frac{d}{d\lambda} [f'(\lambda_0) - \xi] = \frac{d}{d\lambda} f'(\lambda_0) - \frac{d}{d\lambda} \xi \)

\( \frac{d}{d\lambda} f'(\lambda_0) = f''(\lambda_0) \neq 0 \)

\( \frac{d}{d\lambda} \xi = \frac{d}{d\lambda} [f'(\lambda_0) - \xi] = 0 \)

\( \frac{d}{d\lambda} \xi = 0 \)