MA303 TEST 3 NAME ____________________

Thursday, Nov. 17, 2011

SHOW YOUR WORK. CORRECT WORK AND A CORRECT ANSWER ARE NEEDED TO OBTAIN FULL CREDIT. NO CALCULATORS.

1.(22%) Consider the following matrices:

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 0 \\ 0 & -2 \\ -7 & 3 \end{bmatrix}$$

Compute the following if possible:

(a) $2B$ (b) B^{-1} (c) BC (d) CA .

2.(22%) (a) Find the solution to the following system of algebraic equations in vector form by putting the matrix in RREF.

(b) What is the solution to the corresponding homogeneous system?

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2\,x_1 - x_2 + x_3 &= 2 \\ x_1 + 2\,x_2 - x_3 &= 5 \end{aligned}$$

3.(22%) Find the general solution to the following algebraic system by putting the matrix in RREF. Express the solution in vector form.

$$\begin{aligned} x_1 + x_2 + 2\,x_3 + 2\,x_5 &= 1 \\ x_2 + x_3 + x_5 &= -2 \\ 2\,x_2 + 3\,x_3 + x_4 + 4\,x_5 &= 3 \end{aligned}$$

4.(10%) Give an example of two 2×2 matrices $A \neq O$ and $B \neq O$ but where $A\,B = O$. Verify. O is the matrix with all zero entries.

5.(24%) For each nonhomogeneous differential equation find the general solution $x(t)$.

(a) $\ddot{x} - x = e^t$ (b) $\ddot{x} + 4\,x = \cos(t)$

MA 303 Test 3

1. (a) $2B = 2\begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix} = \begin{bmatrix}2 & 2\\ -4 & -2\end{bmatrix}$

(b) B^{-1} $\quad [B\,|\,I] = \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ -2 & -1 & 0 & 1\end{array}\right] \underset{2r_1+r_2}{\sim} \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\ 0 & 1 & 2 & 1\end{array}\right] \overset{-r_2+r_1}{\sim} \left[\begin{array}{cc|cc}1 & 0 & -1 & -1\\ 0 & 1 & 2 & 1\end{array}\right]$

$B^{-1} = \begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$ $\qquad BB^{-1} = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ ✓

(c) $\underset{2\times 2\ \ 3\times 2}{BC}$ $\qquad$ # col. B ≠ # rows C $\quad$ so *not possible*

(d) $\underset{\substack{3\times 2\ \ 2\times 2\\ \Rightarrow 3\times 2}}{CA} = \begin{bmatrix}4 & 0\\ 0 & -2\\ -7 & 3\end{bmatrix}\begin{bmatrix}2 & -1\\ 4 & 0\end{bmatrix} = \begin{bmatrix}8 & -4\\ -8 & 0\\ \underbrace{-14+12}_{=-2} & 7\end{bmatrix}$

2. (a) $\begin{array}{c}\begin{array}{cccc}x_1 & x_2 & x_3 & \end{array}\\ \left[\begin{array}{ccc|c}1 & -1 & 1 & 0\\ 2 & -1 & 1 & 2\\ 1 & 2 & -1 & 5\end{array}\right]\end{array} \underset{\substack{-2r_1+r_2\\ -r_1+r_3}}{\sim} \left[\begin{array}{ccc|c}1 & -1 & 1 & 0\\ 0 & 1 & -1 & 2\\ 0 & 3 & -2 & 5\end{array}\right] \underset{-3r_2+r_3}{\overset{r_2+r_1}{\sim}} \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & -1 & 2\\ 0 & 0 & 1 & -1\end{array}\right]$

$\underset{r_3+r_2}{\sim} \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\end{array}\right] \Rightarrow \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}2\\ 1\\ -1\end{bmatrix}$ $\qquad$ *unique solution*

(b) homog sol to $A\vec{x} = \vec{0}$ is $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ since RREF $= I$, i.e., A^{-1} exis

3. $\begin{array}{c}\begin{array}{cccccc}x_1 & x_2 & x_3 & x_4 & x_5 & \end{array}\\ \left[\begin{array}{ccccc|c}1 & 1 & 2 & 0 & 2 & 1\\ 0 & 1 & 1 & 0 & 1 & -2\\ 0 & 2 & 3 & 1 & 4 & 3\end{array}\right]\end{array} \underset{-2r_2+r_3}{\overset{-r_2+r_1}{\sim}} \left[\begin{array}{ccccc|c}1 & 0 & 1 & 0 & 1 & 3\\ 0 & 1 & 1 & 0 & 1 & -2\\ 0 & 0 & 1 & 1 & 2 & 7\end{array}\right]$

$\underset{\substack{-r_3+r_1\\ -r_3+r_2}}{\sim} \begin{array}{c}\begin{array}{cccccc} & & & x_4\text{ free} & x_5\text{ free} & \end{array}\\ \left[\begin{array}{ccccc|c}1 & 0 & 0 & -1 & -1 & -4\\ 0 & 1 & 0 & -1 & -1 & -9\\ 0 & 0 & 1 & 1 & 2 & 7\end{array}\right]\end{array} \Rightarrow$

$x_1 = -4 + x_4 + x_5$

$x_2 = -9 + x_4 + x_5$

$x_3 = 7 - x_4 - 2x_5$

$$\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix} = \begin{bmatrix}-4\\ -9\\ 7\\ 0\\ 0\end{bmatrix} + x_4\begin{bmatrix}1\\ 1\\ -1\\ 1\\ 0\end{bmatrix} + x_5\begin{bmatrix}1\\ 1\\ -2\\ 0\\ 1\end{bmatrix}$$

4. Find $A \neq 0$, $B \neq 0$ so $AB = 0$

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} \quad \text{verified}$$

5. (a) $\ddot{x} - x = e^t$

Assume $x^{(h)}(t) = e^{\lambda t}$ to get char. eq:

$\lambda^2 e^{\lambda t} - e^{\lambda t} = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$x^{(h)}(t) = c_1 e^t + c_2 e^{-t}$

Find $x^{(p)}(t)$. Try $x(t) = Ae^t$ Res

Try $x(t) = Ate^t$, $\dot{x} = Ae^t + Ate^t$ and

$\ddot{x} = Ae^t + Ae^t + Ate^t$

In diff eq: $2Ae^t + \cancel{Ate^t} - \cancel{Ate^t} = e^t$

$2Ae^t = e^t \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

Gen sol. $x(t) = c_1 e^t + c_2 e^{-t} + \frac{1}{2} te^t$

(b) $\ddot{x} + 4x = \cos(t)$

ch eq. $\lambda^2 + 4 = 0 \rightarrow \lambda = \pm 2i$ so complex sol. $z(t) = e^{i(2t)}$

Hence $x^{(h)}(t) = c_1 \cos(2t) + c_2 \sin(2t)$

Find $x^{(p)}(t)$. Try $x(t) = A\cos(t) + B\sin(t)$

$\dot{x} = -A\sin t + B\cos t$

$\ddot{x} = -A\cos t - B\sin t$

In diff. eq:

$-A\cos t - B\sin t + 4A\cos t + 4B\sin t = \cos t$

$3A\cos t + 3B\sin t = \cos t$

At $t = 0$: $3A = 1 \rightarrow A = \frac{1}{3}$, At $t = \frac{\pi}{2}$: $3B = 0 \Rightarrow B = 0$

Gen sol. $x(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{3}\cos(t)$