Evaluate the integral $\int \frac{1}{x^2\sqrt{x^2+9}}\,dx$.

Have $\sqrt{x^2+9} \rightsquigarrow$ use trig subst $\quad x = 3\tan\theta$

$\sqrt{x^2+9} = 3\sqrt{\tan^2\theta+1} = 3\sec\theta \qquad dx = 3\sec^2\theta\,d\theta$

THEN

$$\int \frac{1}{x^2\sqrt{x^2+9}}\,dx = \int \frac{1}{\underbrace{9\tan^2\theta}_{x^2}\cdot\underbrace{(\cancel{3}\sec\theta)}_{\sqrt{x^2+9}}}\cdot\underbrace{\cancel{3}\sec^{\cancel{2}}\theta\,d\theta}_{dx} =$$

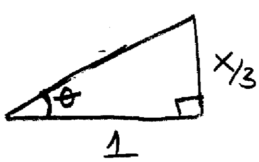
$$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{\cancel{\cos\theta}}\cdot\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\boxed{u = \sin\theta \Rightarrow du = \cos\theta\,d\theta}$$

$$= \frac{1}{9}\int \frac{1}{u^2}\,du = \frac{1}{9}\cdot -\frac{1}{u} + C = -\frac{1}{9\sin\theta} + C \qquad (*)$$

Question: If $x = 3\tan\theta$ What is $\sin\theta$ as a fction of x?

$\frac{x}{3} = \tan\theta$



$\leftarrow$ Build a triangle like this!!! $\qquad \tan\theta = \frac{\frac{x}{3}}{1} = \frac{x}{3}$

then $\sin\theta = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{\frac{x}{3}}{\sqrt{1^2+\frac{x^2}{3^2}}} = \frac{x}{3}\cdot\frac{3}{\sqrt{x^2+9}}$

So $\boxed{\sin\theta = \frac{x}{\sqrt{x^2+9}}}$ $\rightarrow$ substitute this in $(*)$

And finally $\boxed{\int\frac{1}{x^2\sqrt{x^2+9}}\,dx = -\frac{1}{9}\,\frac{\sqrt{x^2+9}}{x} + C}$