Name: ______________________________

Use of books, notes or calculators is **NOT** permitted.

Please show all your work! Answers without appropriate supporting work may not receive full credit.

Clearly indicate your answers to each problem by underlining them or placing a box around your answers!

Trigonometric functions at the values $0, \pi/6, \pi/4, \pi/3, \pi/2$, etc must be evaluated!

T/F Questions are graded with NO PARTIAL CREDIT.

Exam Score

Problem	Score	Out of:
1		10
2		20
3		20
4		10
5		20
6		20
7		5
Total		105

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1. [**10**] For the True/False questions below, clearly circle your answer.

T or [✓ F] If f and g are increasing functions on an interval I, then $f - g$ is increasing on I.

T or [✓ F] If f and g are increasing functions on an interval I then fg is increasing on I.

[✓ T] or F If f is increasing and $f(x) > 0$ on I, then $g(x) = 1/f(x)$ is decreasing on I.

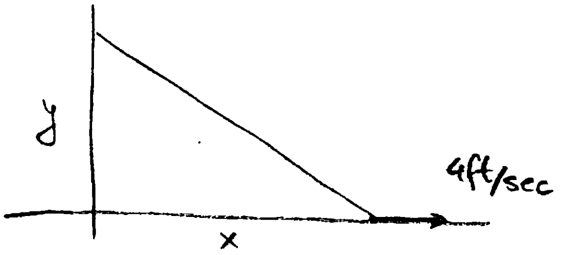
T or [✓ F] If f has an absolute minimum value at c, then $f'(c) = 0$.

[✓ T] or F There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

2. [**20**] A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 4 feet per second. How fast is the top moving down when the base of the ladder is 15 feet from the wall?

$x(t)$ = distance of base of ladder from the wall

$y(t)$ = height of top of ladder



$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x = 15 \Rightarrow y = \sqrt{25^2 - 15^2} = \sqrt{400} = 20 \text{ ft}$$

$$2 \cdot 15 \text{ ft} \cdot 4 \text{ ft/sec} + 2 \cdot 20 \text{ ft} \cdot \frac{dy}{dt} = 0$$

$$\Downarrow$$

$$\frac{dy}{dt} = -\frac{120}{40} = -3 \text{ ft/sec}$$

$$\boxed{\frac{dy}{dt} = -3 \text{ ft/sec}}$$

3. **[20]** Find the following limits. Verify if L'Hospital Rule applies, before using it.

(a) [5 pts]

$$\lim_{x\to 0}\frac{1-\cos x}{e^x} = \frac{1-1}{1} = \frac{0}{1} = \boxed{0}$$

No need for L'Hopital Rule

(b) [5 pts]

$$\lim_{x\to 0^+}\sqrt{x}\ln x = 0\cdot(-\infty)$$

→ Transform and apply l'Hopital

$$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}} \overset{\text{l'Hop}}{=} \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+}\frac{1}{x}\cdot(-2)\cdot x^{3/2} =$$

$$= \lim_{x\to 0^+}(-2)\cdot x^{1/2} = \boxed{0}$$

(c) [5 pts]

$$\lim_{x\to\infty} x^2 e^{-x^2} = \infty\cdot 0$$

Transform and apply l'Hopital

$$\lim_{x\to\infty}\frac{x^2}{e^{x^2}} = \lim_{x\to\infty}\frac{\cancel{2x}}{e^{x^2}\cdot\cancel{2x}} = \lim_{x\to\infty}\frac{1}{e^{x^2}} = \boxed{0}$$

(d) [5 pts]

$$\lim_{x\to\infty}(1+2x)^{\frac{1}{x}}$$

$$y = (1+2x)^{\frac{1}{x}} \qquad \ln y = \frac{1}{x}\ln(1+2x)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1+2x)}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{\frac{1}{1+2x}\cdot 2}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} y = e^0 = \boxed{1}$$

4. [**10**] Use a linear approximation to estimate $(2.01)^5$

$f(x) = x^5$

$f'(x) = 5x^4$

$a = 2$

$f(x) \approx f(a) + f'(a)(x-a)$

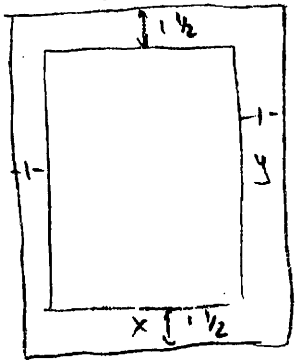
$\approx f(2) + f'(2)(x-2)$

$\approx 32 + 80(x-2)$

$(2.01)^5 = f(2.01) \approx 32 + 80 \cdot 0.01 = 32.8$

5. [**20**] A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

(a) [3 pts] Draw a picture of the rectangular page and its printed area. Label with x and y the dimensions of the printed area.



(b) [3 pts] Express the area to be minimized as a function of x and y.

$A(x,y) = (x+2)(y+3)$

(c) [3 pts] Express the area of print as a function of x and y.

$P = xy$

(d) [3 pts] Express the area to be minimized as a function of x *only*. Use the equations you found in (b) and (c) to do this.

$P(x,y) = xy = 24 \Rightarrow y = \frac{24}{x}$

$$A(x) = (x+2)\left(\frac{24}{x}+3\right) = 24+3x+\frac{48}{x}+6 = 3x+\frac{48}{x}+30$$

(e) [8 pts] Find the dimensions of the page which minimize the amount of paper used. Place your answers (length, width) in a box.

Want to minimize $A(x) = 3x+\frac{48}{x}+30$

Feasible region $\begin{matrix} x>0 \\ y>0 \end{matrix}$ | they are lengths of the page!

$A'(x) = 3-\frac{48}{x^2}$ $\quad A'(x)=0 \Leftrightarrow 3-\frac{48}{x^2}=0 \Leftrightarrow x^2=16 \Leftrightarrow$

$$\left.\begin{matrix} x=\pm 4 \\ x>0 \end{matrix}\right\} \Rightarrow \boxed{x=4}$$

Check if $x=4$ is a min → 2nd deriv test

$$\left.\begin{matrix} A''=+\frac{96}{x^3} \\ A''(4)=\frac{96}{64}>0 \end{matrix}\right\} \Rightarrow \underline{x=4 \text{ is a minimum point}}$$

$x=4 \Rightarrow y=\frac{24}{4}=6$

The dimensions of the page are

$$\boxed{\begin{matrix} x+2 = 6 \text{ in} \\ y+3 = 9 \text{ in} \end{matrix}}$$

6. [**20**] Let us consider the function $f(x) = x^3 - 3x^2 + 1$ on the closed interval $[-1, 4]$.

(a) [10 pts] Find the absolute maximum and minimum values of f and state where those values occur.

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0 \iff 3x(x-2) \Rightarrow \begin{matrix} x=0 \\ x=2 \end{matrix} >$ critical #s in $[-1,4]$

f @ critical #s $f(0) = 1$

$f(2) = -3 \leftarrow$ ABS MIN

f @ endpoints $f(-1) = -3 \leftarrow$ ABS MIN

$f(4) = 17 \leftarrow$ ABS MAX

ABS MAX = 17 occurring at $x=4$

ABS MIN = -4 occurring at $x=-1$ and $x=2$

(b) [10 pts] On what subinterval of $[-1, 4]$ is f increasing/decresing? On what subinterval of $[-1, 4]$ is f concave up/down? Sketch the graph of f.

f increasing if $f'(x) = 3x(x-2) > 0 \iff$ outside the roots $\iff x \in (-1,0) \cup (2,4)$

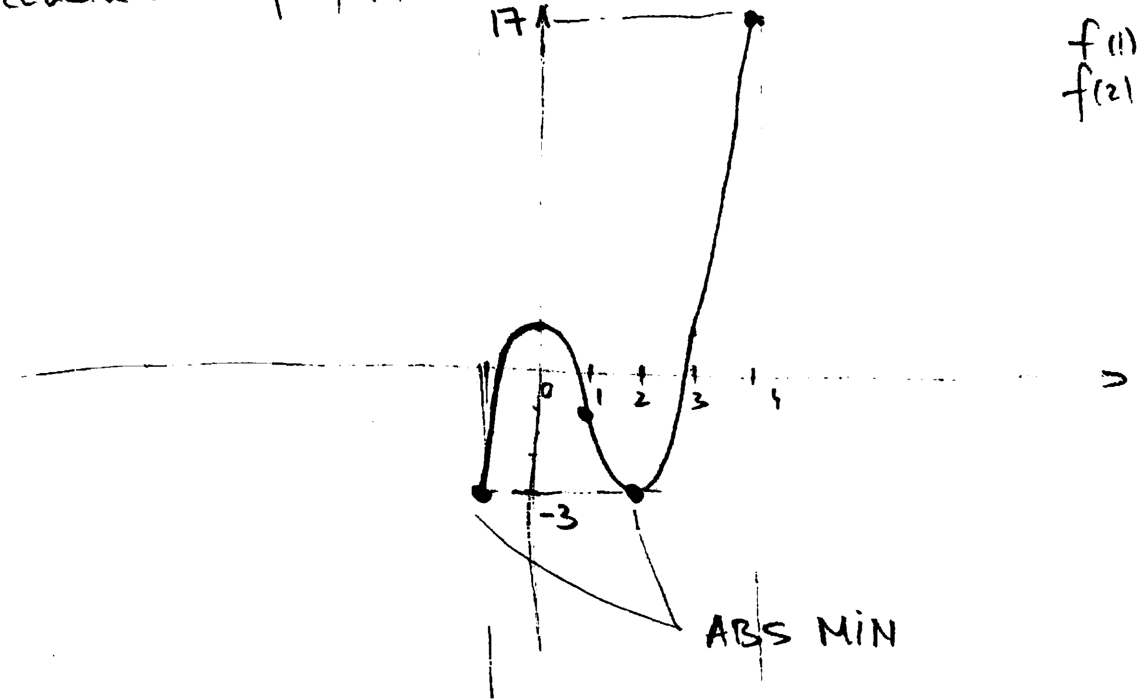
f decreasing if $f'(x) = 3x(x-2) < 0 \iff$ inside the roots $\iff x \in (0,2)$

f concave up if $f''(x) = 6x-6 > 0 \iff x > 1 \rightarrow x \in (1,4)$

f concave down if $f''(x) = 6x-6 < 0 \iff x < 1 \rightarrow x \in (-1,1)$

$f(1) = -1$

$f(2) = -3$



7. [**5 Bonus Points**] Calculate up to four decimal places x_3, the third iterate of Newton's Method which approximates a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as initial guess. You may use the fact that $0.25/3 = 0.0833$.

$f(x) = x^2 - 2$

$f'(x) = 2x$

$f(1) = -1$

$f'(1) = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{-1}{2} = 1.5$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_1)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{2.25}{3} =$$

$$= 1.5000 - 0.0833 = 1.4167$$

$$\boxed{x_3 = 1.4167}$$