Name: ______________________________

Use of books, notes or calculators is **NOT** permitted.

Please show all your work! Answers without appropriate supporting work may not receive full credit.

Clearly indicate your answers to each problem by underlining them or placing a $\boxed{\text{box}}$ around your answers!

Trigonometric functions at the values $0, \pi/6, \pi/4, \pi/3, \pi/2$, etc must be evaluated!

T/F Questions are graded with NO PARTIAL CREDIT.

Exam Score

Problem	Score	Out of:
1		10
2		20
3		20
4		10
5		20
6		20
7		5
Total		105

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1. [**10**] For the True/False questions below, clearly circle your answer.

T or F If f and g are increasing functions on an interval I, then $f+g$ is increasing on I.

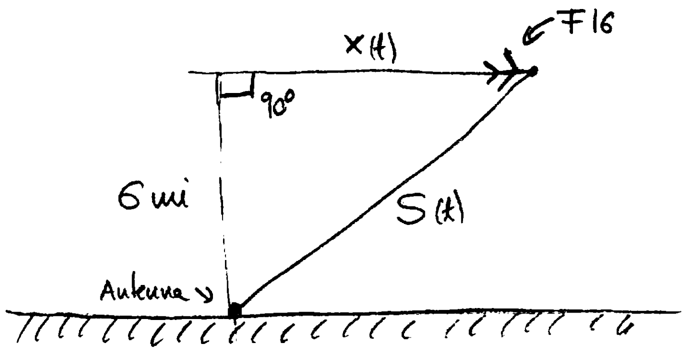
T or F If f and g are positive increasing functions on an interval I then fg is increasing on I.

T or F If $f''(2) = 0$ then $(2, f(2))$ is an inflexion point of the curve $y = f(x)$.

T or F If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

T or F If f is differentiable and $f(0) = f(1)$, then there is a number c in $(0,1)$ such that $f'(c) = 0$.

2. [**20**] An F16 fighter jet is flying at an altitude of 6 miles and passes directly over a radar antenna. When the plane is 10 miles away from the antenna, the radar detects that the distance between the antenna and the plane is changing at a rate of 480 mi/h. What is the speed of the aircraft?



x(t) = position of F16

S(t) = distance between antena and F16

$$x^2 + 6^2 = s^2 \quad \Big| \frac{d}{dt}$$

$$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$$

When $s = 10$ mi $\quad x = \sqrt{10^2 - 6^2} = \sqrt{100-36} = \sqrt{64} = 8$ mi

So :

$$\frac{dx}{dt}\Big|_{\text{when } s = 10\text{ miles}} = \frac{\cancel{2}s \cdot \frac{ds}{dt}}{\cancel{2}x} = \frac{10\text{ mi } 480 \text{ mi/h}}{8 \text{ mi}} = 600 \text{ mi/h}$$

$$\boxed{\text{Speed} = 600 \text{ mi/h}}$$

3. [**20**]Find the following limits. Verify if L'Hospital Rule applies, before using it.

(a) [5 pts]

$$\lim_{x\to 0} \frac{e^{2x}-e^{3x}}{\sec x} = \frac{1-1}{1} = \frac{0}{1} = \boxed{0}$$

No need for L'Hospital Rule !

(b) [5 pts]

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = [0\cdot\infty] \to$$ Transform to apply L'Hospital

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^x} = \left[\frac{\infty}{\infty}\right] \overset{\text{L'Hospital}}{=} \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}\,e^x} = \boxed{0}$$

(c) [5 pts]

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right) = \infty\cdot\tan\left(\frac{1}{\infty}\right) = \infty\cdot\tan 0 = \infty\cdot 0$$

Transform and apply L'Hospital

$$\lim_{x\to\infty} x\tan\frac{1}{x} = \lim_{x\to\infty}\frac{\tan\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\sec^2\frac{1}{x}\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty}\sec^2\frac{1}{x} = \boxed{1}$$

(d) [5 pts]

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} \qquad 1^\infty$$

$$y = \left(1+\frac{1}{x}\right)^{2x} \qquad \ln y = 2x\ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} 2x\ln\left(1+\frac{1}{x}\right) = [\infty\cdot 0] = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{2x}} =$$

$$\overset{\text{L'Hospital}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\cancel{\frac{1}{x^2}}\right)}{\frac{1}{2}\cdot\left(-\cancel{\frac{1}{x^2}}\right)} = \lim_{x\to\infty}\frac{2}{1+\frac{1}{x}} = 2$$

So $\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x} = e^2}$

4. [**10**] Use a linear approximation to estimate $\sqrt{36.24}$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$f(x) \approx f(36) + f'(36)(x-36)$

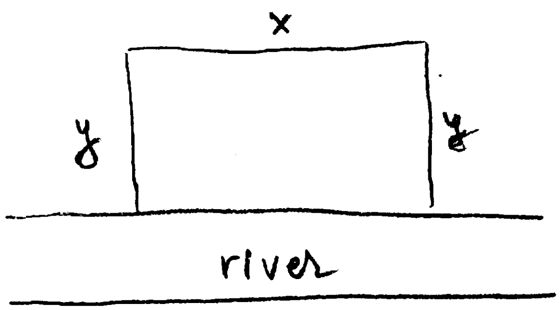
$\approx \sqrt{36} + \frac{1}{2\sqrt{36}}(x-36)$

$\approx 6 + \frac{1}{12}(x-36)$

$\sqrt{36.24} \approx 6 + \frac{1}{12}(36.24-36) \approx 6 + \frac{1}{12} \cdot 0.24 \approx 6.20$

5. [**20**] A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1800 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

(a) [3 pts] Draw a picture of the rectangular pasture and the river and label on it the sides of interest. Use x for the side parallel to the river, and y for the sides perpendicular to the river.



(b) [3 pts] Express the amount of fencing as a function of x and y.

$f(x,y) = x + 2y$

amount of fencing

(c) [3 pts] Express the area of pasture as a function of x and y.

Area $= xy$

(d) [3 pts] Express the amount of fencing as a function of x *only*. Use the equations you found in (b) and (c) to do this.

$$\text{Area} = xy = 1800 \Rightarrow y = \frac{1800}{x}$$

$$f(x) = x + 2 \cdot \frac{1800}{x} = x + \frac{3600}{x}$$

(e) [8 pts] Find the dimensions of the rectangular pasture with least fencing. Place your answers (length, width) in a box.

Want to minimize $f(x) = x + \frac{3600}{x}$

Feasible domain $x>0$, $y>0$ because they are lengths !!!

$$f'(x) = 1 - \frac{3600}{x^2} \qquad f'(x)=0 \Leftrightarrow 1-\frac{3600}{x^2}=0 \Leftrightarrow x^2=3600$$

$$\Leftrightarrow x = \pm 60$$

Because $x>0$, $y>0$ only $\boxed{x=60}$ is in the feasible domain

Check if $x=60$ is a min point $\to$ 2nd derivative test

$$f''(x) = \frac{7200}{x^3}$$

$$f''(60) = \frac{7200}{60^3} = \frac{2}{60} = \frac{1}{30} > 0 \Rightarrow \boxed{x=60 \text{ is a min point}}$$

$$y = \frac{1800}{60} = 30$$

$$\boxed{\begin{array}{l} x = 60 \text{ m} \\ y = 30 \text{ m} \end{array}}$$

6. [**20**] Let us consider the function $f(x) = x^3 - 3x^2$ on the closed interval $[-1, 4]$.

(a) [10 pts] Find the absolute maximum and minimum values of f and state where those values occur.

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x) = 0 \Leftrightarrow 3x(x-2) = 0 \Rightarrow x = 0, \; x = 2$ > critical #s in $[-1,4]$

f@ critical #s $f(0) = 0$
$f(2) = -4$ ← ABS MIN

f@ end points $f(-1) = -4$ ← ABS MIN
$f(4) = 16$ ← ABS MAX

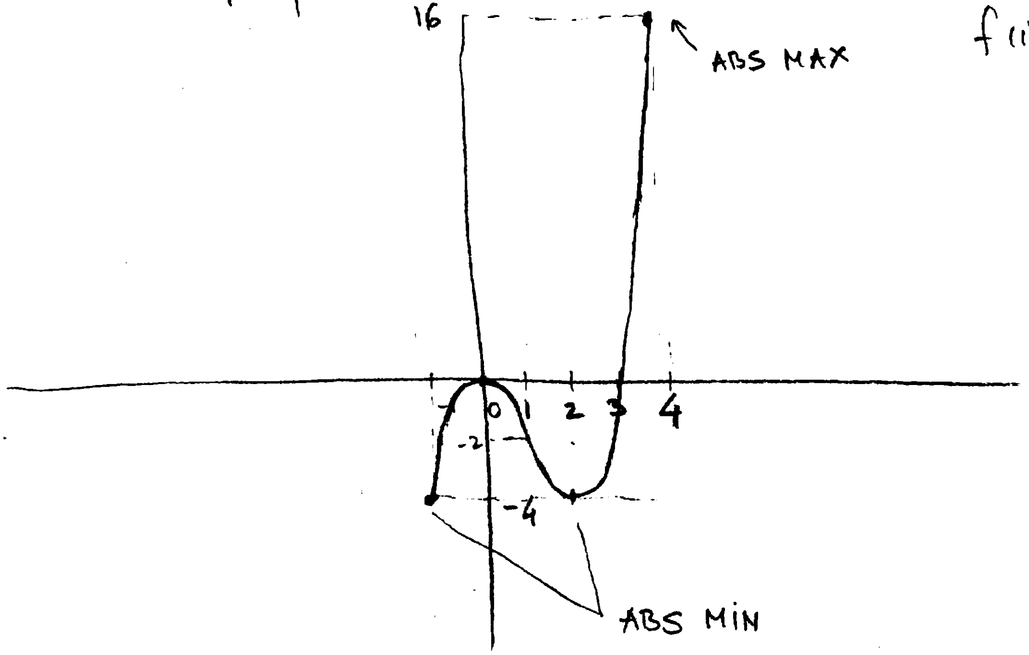
$\boxed{\text{ABS MAX} = 16 \text{ occuring at } x = 4 \quad \text{ABS MIN} = -4 \text{ occuring at } x = -1 \text{ and } x = 2}$

(b) [10 pts] On what subinterval of $[-1, 4]$ is f increasing/decresing? On what subinterval of $[-1, 4]$ is f concave up/down? Sketch the graph of f.

f increasing if $f'(x) = 3x(x-2) > 0 \Leftrightarrow$ outside the roots $\Leftrightarrow x \in (-1,0) \cup (2,4)$
f decreasing if $f'(x) = 3x(x-2) < 0 \Leftrightarrow$ inside the roots $\Leftrightarrow x \in (0,2)$

f concave up if $f''(x) = 6x - 6 > 0 \Leftrightarrow x > 1 \rightarrow x \in (1,4)$
f concave down if $f''(x) = 6x - 6 < 0 \Leftrightarrow x < 1 \rightarrow x \in (-1,1)$

$f(1) = -2$



7. [**5 Bonus Points**] Calculate up to four decimal places x_3, the third iterate of Newton's Method which approximates a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as initial guess. You may use the fact that $0.25/3 = 0.0833$.

$$\boxed{\begin{array}{l} f(x) = x^2-2 \\ f'(x) = 2x \end{array}}$$

$f(1) = -1$

$f'(1) = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{-1}{2} =$$

$$= 1.5 \checkmark$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{2.25}{3} =$$

$$= 1.5000 - 0.0833 = 1.4167 \checkmark$$

$$\begin{array}{r} 1.5000 - \\ 0.0833 \\ \hline 1.4167 \end{array}$$

$$\boxed{x_3 = 1.4167}$$