Name: ____________________

Use of books, notes or calculators is **NOT** permitted.

Please show all your work! Answers without appropriate supporting work may not receive full credit.

Clearly indicate your answers to each problem by underlining them or placing a box around your answers!

Trigonometric functions at the values $0, \pi/6, \pi/4, \pi/3, \pi/2$, etc must be evaluated!

T/F Questions are graded with NO PARTIAL CREDIT.

There is a total of 4 **DOUBLE-SIDED** pages to this exam including the cover page.

Solutions ~~to~~ Test 2B

Exam Score

Problem	Score	Out of:
1		10
2		10
3		20
4		20
5		25
6		10
7		10
Total		105

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1. [**10**] For the True/False questions below, clearly circle your answer.

T or F If f is differentiable at a, then f is continuous at a.

T or F $\frac{d}{dx}e^2 = 2e$.

T or F If $f(x) = x^3$, then $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 3$.

T or F If f is differentiable, and $f' < 0$ on (a, b), then f is increasing on (a, b).

T or F If f and g are differentiable, then $[f(g(x))]' = f'(g(x)) \cdot g'(x)$.

2. [**10**] Use the limit definition of derivative to find the derivative of $f(x) = x^2 - x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-x} - h \cancel{-x^2} \cancel{+x}}{h} =$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - h}{h} = \lim_{h\to 0} (2x + h - 1) =$$

$$= 2x - 1$$

$$\boxed{\text{So: } f'(x) = 2x - 1}$$

3. [**20**]Find the indicated derivatives.

(a) [5 pts] $f(x) = x^2 e^{x^2-1}$. Find $f'(1)$

$$f'(x) = 2x\, e^{x^2-1} + x^2 \cdot e^{x^2-1} \cdot 2x$$

$$f'(1) = 2e^0 + e^0 \cdot 2 = 4$$

(b) [5 pts] $g(y) = \frac{1}{\sqrt[3]{y}} - \sin \pi y + \ln \sqrt{y}$, find $\frac{dg}{dy}$

$$g(y) = y^{-\frac{1}{3}} - \sin \pi y + \frac{1}{2} \ln y$$

$$g'(y) = -\frac{1}{3} y^{-\frac{4}{3}} - \pi \cos \pi y + \frac{1}{2} \cdot \frac{1}{y}$$

(c) [5 pts] $h(t) = \frac{t^2-1}{t+2}$. Find $h'(t)$

$$h'(t) = \frac{2t \cdot (t+2) - (t^2-1) \cdot 1}{(t+2)^2} = \frac{2t^2 + 4t - t^2 + 1}{(t+2)^2} =$$

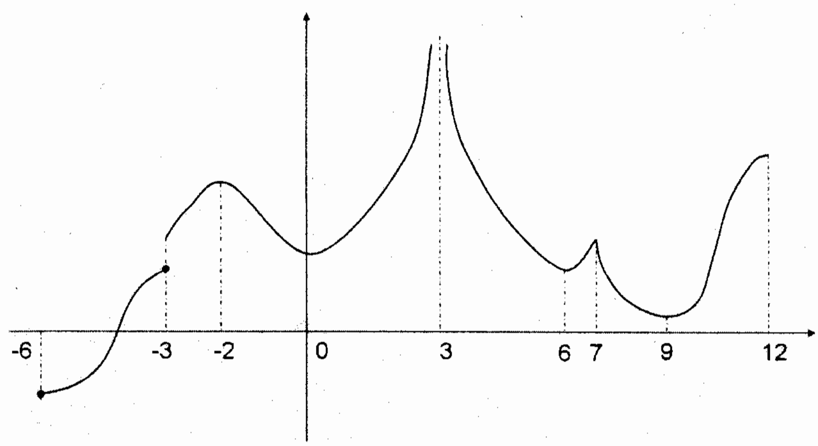
$$= \frac{t^2+4t+1}{(t+2)^2}$$

(d) [5 pts] $r(s) = (1+s^2) \arctan s$. Find $\frac{dr}{ds}$

$$r'(s) = 2s \arctan(s) + \cancel{(1+s^2)} \cdot \frac{1}{\cancel{1+s^2}}$$

$$= 2s \arctan(s) + 1$$

4. [**20**] The graph of the function $f : [-6, 12) \to \mathbf{R}$ is shown below.



(a) [5 pts] State with reasons the points at which f is not differentiable.

-3 (f is not even continuous at this point)

3 (f is not even continuous at this point

7 ($f'(7)$ does not exist. $f'_\ell(7) \neq f'_r(7)$)

(b) [5 pts] State with reasons the intervals where $f' > 0$.

$(-6,-3)$, $(-3,-2)$, $(0,3)$, $(6,7)$, $(9,12)$

Because f is increasing on these intervals.

(c) [5 pts] State with reasons the intervals where $f' < 0$.

$(-2,0)$, $(3,6)$, $(7,9)$

Because f is decreasing on these intervals

(d) [5 pts] State with reasons the points where $f' = 0$.

$-2, 0, 6, 9$ Because the tangent line is horizontal at these points.

5. [**25**] Let us consider the function $f(x) = x^3 + 3x^2 - 24x - 6$.

(a) [10 pts] On what interval is f decreasing?

A diff function is increasing on a interval if and only if $f'(x) > 0$ on that interval.

$$f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x+4)(x-2)$$

$$f'(x) > 0 \iff 3(x+4)(x-2) > 0$$

x	$-\infty$		-4		2		$+\infty$
$x+4$		$- - - -$	0	$+ + + +$	$+$	$+ + +$	
$x-2$		$- - - -$	$-$	$- - -$	0	$+ + + + +$	
$(x+4)(x-2)$		$+ + +$	0	$- - -$	0	$+ + + + +$	

So $f'(x) > 0$ for

$$\boxed{x \in (-\infty, -4) \cup (2, +\infty)}$$

(b) [10 pts] On what interval is f concave downward?

A diffb fcn $f(x)$ is concave downward on an interval if and only if $f''(x) < 0$ on that interval.

$$f''(x) = (3x^2 + 6x - 24)' = 6x + 6 = 6(x+1)$$

$$f''(x) < 0 \iff 6(x+1) < 0 \iff x < -1$$

$$\boxed{\text{So } f \text{ is concave downwards on } (-\infty, -1) \text{ or for } x < -1}$$

(c) [**5 Extra Bonus** pts] Show that f has a solution in the interval $(-1, 0)$.

f is a continuous fcn on the interval $[-1, 0]$ (Why? Because is a polynomial)

$$f(-1) = -1 + 3 + 24 - 6 = 20$$
$$f(0) = -6$$

So: $f(-1)f(0) < 0 \Rightarrow$ According to the Intermediate Value Theorem the result follows.

6. [**10**] Use implicit differentiation to find an equation of the tangent line to the hyperbola $x^2 + 2xy - y^2 + x = 2$ at the point $(1, 2)$.

Eq of tangent line is $y - 2 = m(x-1)$

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 1 = 0$$

$$\frac{dy}{dx} = \frac{-2x-2y-1}{2x-2y}$$

At $(1,2)$ $\quad m = \frac{dy}{dx} = \frac{-2\cdot1-2\cdot2-1}{2\cdot1-2\cdot2} = \frac{-7}{-2} = \frac{7}{2}$

So eq of tang line is $\boxed{y-2 = \frac{7}{2}(x-1) \quad \text{or} \quad y = \frac{7}{2}x - \frac{3}{2}}$

7. [**10**] Use logarithmic differentiation to find the derivative of the function $y = (\cos x)^x$.

$$y = (\cos x)^x$$

$$\ln y = \ln(\cos x)^x = x \ln \cos x \quad \Big| \frac{d}{dx}$$

$$\frac{y'}{y} = \ln(\cos x) + x \cdot \frac{1}{\cos x} \cdot (-\sin x)$$

$$y' = (\cos x)^x \left[\ln(\cos x) - x\frac{\sin x}{\cos x} \right] =$$

$$= (\cos x)^x \left[\ln(\cos x) - x \tan x \right]$$

$$\boxed{y' = (\cos x)^x \left[\ln(\cos x) - x \tan x \right]}$$