Name: ______________________________

Use of books, notes or calculators is **NOT** permitted.

Please show all your work! Answers without appropriate supporting work my not receive full credit.

Clearly indicate your answers to each problem by underlining them or placing a $\boxed{\text{box}}$ around your answers!

Trigonometric functions at the values $0, \pi/6, \pi/4, \pi/3, \pi/2$, etc must be evaluated!

T/F Questions are graded with NO PARTIAL CREDIT.

Solutions to Practice Test 2

Exam Score

Problem	Score	Out of:
1		10
2		10
3		20
4		20
5		25
6		10
7		10
Total		105

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1. [10] For the True/False questions below, clearly circle your answer.

T or F — If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

T or F — An equation of the tangent line to the parabola $y = x^2$ at $(2, 4)$ is $y - 4 = 2x(x - 2)$.

T or F — $\frac{d}{dx}(\ln 10) = \frac{1}{10}$.

T or F — $\frac{d}{dx}(10^x) = x10^{x-1}$.

T or F — If f and g are differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

2. [10] Use the limit definition of derivative to find the derivative of $f(x) = 3x^2 + x$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h)^2 + (x+h) - (3x^2+x)}{h} =$$

$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2) + x + h - 3x^2 - x}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{x} + h - \cancel{3x^2} \cancel{-x}}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 + h}{h} = \lim_{h\to 0} (6x + 3h + 1) =$$

$$= 6x + 1$$

So $\boxed{f'(x) = 6x + 1}$

3. [**20**]Find the indicated derivatives.

(a) [5 pts] $f(x) = e^{x^2-1} + \frac{4}{x} + 6x^{\frac{2}{3}} - \pi^2$. Find $f'(1)$

$$f'(x) = 2x\,e^{x^2-1} - \frac{4}{x^2} + 6\cdot\frac{2}{3}\,x^{-1/3} - 0 = 2xe^{x^2-1} - \frac{4}{x^2} + 4x^{-1/3}$$

$$f'(1) = 2\cdot e^0 - \frac{4}{1} + 4\cdot 1^{-1/3} = 2 - 4 + 4 = 2$$

$$\boxed{f'(1) = 2}$$

(b) [5 pts] $g(y) = \frac{1}{y^2} + \ln\sqrt{y}$, find $\frac{d^2g}{dy^2}$

$$g'(y) = -2y^{-3} + \frac{1}{\sqrt{y}}\cdot\frac{1}{2\sqrt{y}} = -2y^{-3} + \frac{1}{2}y^{-1}$$

$$g''(y) = 6y^{-4} - \frac{1}{2}y^{-2} = \boxed{\frac{6}{y^4} - \frac{1}{2y^2}}$$

(c) [5 pts] $h(t) = t\arctan(t^2)$. Find $h'(t)$

$$h'(t) = \arctan(t^2) + t\cdot\frac{1}{1+(t^2)^2}\cdot 2t =$$

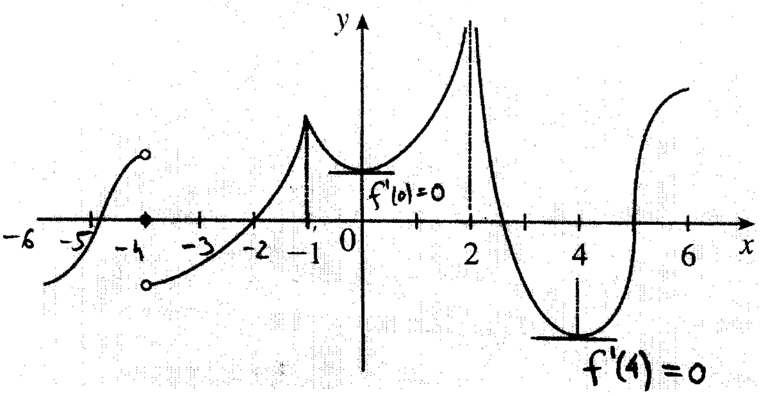
$$= \boxed{\arctan(t^2) + \frac{2t^2}{1+t^4}}$$

(d) [5 pts] $r(s) = \left(\frac{s}{s^2+1}\right)^2$. Find $\frac{dr}{ds}$

$$r'(s) = 2\left(\frac{s}{s^2+1}\right)\cdot\left(\frac{s}{s^2+1}\right)' = \frac{2s}{s^2+1}\cdot\frac{1(s^2+1) - s\cdot 2s}{(s^2+1)^2} =$$

$$= \frac{2s}{s^2+1}\cdot\frac{1-s^2}{(s^2+1)^2} = \boxed{\frac{2s(1-s^2)}{(s^2+1)^3}}$$

4. [**20**] The graph of the function f is shown below.



(a) [5 pts] State with reasons the points at which f is not differentiable.

$\boxed{f \text{ is not differentiable at } -4,\ -1,\ \text{and } 2.}$

f is not diffb at −4 and 2 because f is not even continuous at these points. (Recall f diffb at $x_0 \Rightarrow$ f is continuous at x_0)
f is not diffb at −1 because the graph of f does not have a tangent line at this point ($f'_\ell(-1) \neq f'_r(-1)$)

(b) [5 pts] State with reasons the intervals where $f' > 0$.

$\boxed{f' > 0 \text{ on } (-6,-4),\ (-4,-1),\ (0,2),\ (4,6)}$

since f is increasing on these intervals!

(c) [5 pts] State with reasons the intervals where $f' < 0$.

$\boxed{f' < 0 \text{ on } (-1,0) \text{ and on } (2,4)}$

since f is decreasing on these intervals!

(d) [5 pts] State with reasons the points where $f' = 0$.

$\boxed{x=0}$ and $\boxed{x=4}$ since the slope of the tangent line is zero at these points!

5. [**25**] Let us consider the function $f(x) = \frac{\ln x}{x}$.

(a) [**5 Extra Bonus** pts] What is the domain of f?

$$\left.\begin{array}{l}\ln x \text{ is defined on } (0,+\infty) \\ x \neq 0 \text{ (to avoid division by 0)}\end{array}\right\} \Rightarrow \boxed{D(f) = (0,+\infty)}$$

(b) [10 pts] On what interval is f increasing?

f is increasing when $f'(x) > 0$

$$f'(x) = \frac{[\ln(x)]' \cdot x - \ln(x) \cdot 1}{x^2} = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f'(x) > 0 \Leftrightarrow \frac{1 - \ln x}{x^2} > 0 \Leftrightarrow 1 - \ln x > 0 \quad (\text{since } x^2 > 0)$$

$$\Leftrightarrow \ln x < 1 \;|\, e \Leftrightarrow x < e^1$$

$$\left.\begin{array}{l}\text{Since } x > 0 \text{ (domain of } f \text{ was } (0,+\infty)) \\ x < e\end{array}\right\} \Rightarrow \boxed{f \text{ is increasing on } (0,e)}$$

(c) [10 pts] On what interval is f concave upward?

f is concave upward on intervals where $f''(x) > 0$

$$f''(x) = \frac{(1-\ln x)' \cdot x^2 - (1-\ln x) \cdot 2x}{x^4} = \frac{-\frac{1}{x} \cdot x^2 - 2x + 2x \ln x}{x^4} =$$

$$= \frac{-3x + 2x \ln x}{x^4} = \frac{2 \ln x - 3}{x^3}$$

$$\left.\begin{array}{l} f''(x) > 0 \Leftrightarrow \frac{2 \ln x - 3}{x^3} > 0 \\ \text{But } x > 0 \end{array}\right\} \Leftrightarrow 2 \ln x - 3 > 0 \Leftrightarrow$$

$$\ln x > \frac{3}{2} \Leftrightarrow \underline{x > e^{3/2}}$$

So $\boxed{f \text{ is concave upwards on } (e^{3/2}, +\infty)}$

6. [**10**] Use implicit differentiation to find an equation of the tangent line to the cardioid $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at the point $(0, 1/2)$.

$$2x + 2yy' = 2(2x^2 + 2y^2 - x) \cdot (4x + 4yy' - 1)$$

Substitute $x = 0$ and $y = 1/2$ above ~~to~~ to get y' = the slope of the tangent line at $(0, 1/2)$

$$2 \cdot 0 + 2 \cdot \frac{1}{2} y' = 2\left(2 \cdot 0^2 + 2\left(\frac{1}{2}\right)^2 - 0\right) \cdot \left(4 \cdot 0 + 4 \cdot \frac{1}{2} y' - 1\right)$$

$$y' = 2 \cdot \frac{1}{2} \cdot (2y' - 1) \iff$$

$$y' = 2y' - 1 \iff y' - 2y' = -1 \iff \boxed{y' = 1}$$

So the slope is $y' = 1$. Then, the eq of tang line is

$$y - \frac{1}{2} = 1 \cdot (x - 0) \iff \boxed{y = x + \frac{1}{2}}$$

7. [**10**] Use logarithmic differentiation to find the derivative of the function $y = x^{\sqrt{x}}$.

$$y = x^{\sqrt{x}} \iff \ln y = \sqrt{x} \ln x \quad \Big| \frac{d}{dx}$$

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}$$

So $$y' = y\left(\frac{1}{2\sqrt{x}} \ln x + \frac{\overset{2}{1}}{\sqrt{x}}\right) =$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

$$\boxed{y' = x^{\sqrt{x}}\left(\frac{2 + \ln(x)}{2\sqrt{x}}\right)}$$