1. Let $B$ be an invertible $n \times n$ matrix, and let $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be $C^1$ with $\sup_x \|DG(x)\| < \frac{1}{\|B^{-1}\|}$. Define $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $F(x) = Bx + G(x)$. Prove: $F$ has a $C^1$ inverse.

Method: We want to solve the equation $F(x) = y$ for $x$ in terms of $y$. Rewrite this equation as $Bx + G(x) = y$, then as $x = B^{-1}(y - G(x))$. Define $T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $T(x, y) = B^{-1}(y - G(x))$. We have shown that for a given $y \in \mathbb{R}^n$, $F(x) = y$ if and only if $x = T(x, y)$, i.e., if and only if $x$ is a fixed point of $T(\cdot, y)$. Now you can use the Contraction Mapping Theorem with Parameters to prove the result. You will need to show that for each $y \in \mathbb{R}^n$, $T(\cdot, y)$ is a contraction of $\mathbb{R}^n$.

2. **Implicit Function Theorem.** Let $X$ and $Y$ be Banach spaces, let $U$ be an open subset of $X$, and let $V$ be an open subset of $Y$. Let $f : U \times V \rightarrow Y$ be $C^1$. Let $(x_0, y_0) \in U \times V$. Assume that $f(x_0, y_0) = 0$ and that $D_2f(x_0, y_0)$ is invertible. Show that the exist neighborhoods $U_0$ of $x_0$ and $V_0$ of $y_0$ such that for each $x \in U_0$ there is a unique $y \in V_0$ such that $f(x, y) = 0$. Moreover, if we write $y = g(x)$, then $g$ is $C^1$.

Method: Let $A = D_2f(x_0, y_0)$. Define $T : U \times V \rightarrow Y$ by $T(x, y) = y - A^{-1}f(x, y)$. Notice that $T$ is $C^1$.

(a) Show that $T(x, y) = y$ if and only if $f(x, y) = 0$.

Choose $\delta > 0$ such that if $\|x - x_0\| < \delta$ and $\|y - y_0\| \leq \delta$, then $\|D_2f(x, y) - A\| < \frac{1}{2\|A^{-1}\|}$. Now choose $\epsilon$, $0 < \epsilon \leq \delta$, such that if $\|x - x_0\| < \epsilon$, then $\|f(x, y_0)\| \leq \frac{\delta}{2\|A^{-1}\|}$. Let $U_0 = \{x : \|x - x_0\| < \epsilon\}$ and let $V_0 = \{y : \|y - y_0\| \leq \delta\}$.

(b) Show that if $x \in U_0$ and $y \in V_0$ then $T(x, y) \in V_0$. Suggestion:

\[
\|T(x, y) - y_0\| = \|y - A^{-1}f(x, y) - y_0\|
= \|A^{-1}(Ay - f(x, y) - Ay_0 + f(x, y_0)) - A^{-1}f(x, y_0)\|.
\]

(c) Show that if $x \in U_0$ and $y, y' \in V_0$, then $\|T(x, y) - T(x, y')\| \leq \frac{1}{2}\|y - y'\|$. Suggestion:

\[
\|T(x, y) - T(x, y')\| = \|y - A^{-1}f(x, y) - y' + A^{-1}f(x, y')\|
\leq \|A^{-1}\|\|Ay - f(x, y) - Ay' + f(x, y')\|.
\]

(d) Explain how the theorem now follows from the Contraction Mapping Theorem with Parameters.
(e) Now that you know that $g$ is differentiable, use the formula $f(x, g(x)) = 0$ to derive a formula for $Dg(x)$. 