1. A bead on a rotating hoop satisfies the differential equation

\[ \ddot{x} + \dot{x} + \sin x - \mu \sin 2x = 0. \]

Here \( x \) is measured in radians from the bottom of the hoop, and the parameter \( \mu \) is related to the spin rate of the hoop. Letting \( y = \dot{x} \), we obtain the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\sin x + \mu \sin 2x - y.
\end{align*}
\]

(a) Show that for every \( \mu \), \((x, y) = (0, 0)\) is an equilibrium.

(b) Show that the equilibrium at \((x, y) = (0, 0)\) is attracting for \( \mu < \frac{1}{2} \), has a 0 eigenvalue for \( \mu = \frac{1}{2} \), and is not attracting for \( \mu > \frac{1}{2} \).

(c) Use center manifold reduction to show that a pitchfork bifurcation occurs at \( \mu = \frac{1}{2} \). Suggestions:

- Let \( \lambda = \mu - \frac{1}{2} \).
- Let \( y = h(x, \lambda) = x(A + Bx + C\lambda + Dx^2 + \ldots) \). No more terms should be needed.
- Recall that \( \sin x = x - \frac{x^3}{3!} + \ldots \).

(d) Are the new equilibria that appear in the pitchfork bifurcation attracting? (Suggestion: begin by looking at the bifurcation diagram.)