MA 732 Supplementary Problems

April 6, 2004

1. Use center manifold reduction to show that

$$\begin{aligned} \dot{x} &= y - 2x, \\ \dot{y} &= \mu + x^2 - y \end{aligned}$$

has a saddle-node bifurcation at $(x, y, \mu) = (1, 2, 1)$. (Suggestion: Shift coordinates to put this point at $(0, 0, 0)$.)

2. A bead on a rotating hoop satisfies the differential equation

$$\ddot{x} + \dot{x} + \sin x - \mu \sin 2x = 0.$$

Here x is measured in radians from the bottom of the hoop, and the parameter μ is related to the spin rate of the hoop. Letting $y = \dot{x}$, we obtain the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\sin x + \mu \sin 2x - y. \end{aligned}$$

(a) Show that for every μ, $(x, y) = (0, 0)$ is an equilibrium.

(b) Show that the equilibrium at $(x, y) = (0, 0)$ is attracting for $\mu < \frac{1}{2}$, has a 0 eigenvalue for $\mu = \frac{1}{2}$, and is not attracting for $\mu > \frac{1}{2}$.

(c) Use center manifold reduction to show that a pitchfork bifurcation occurs at $\mu = \frac{1}{2}$. Suggestions:

- Let $\lambda = \mu - \frac{1}{2}$.
- Let $y = h(x, \lambda) = x(A + Bx + C\mu + Dx^2 + \ldots)$. No more terms should be needed.
- Recall that $\sin x = x - \frac{x^3}{3!} + \ldots$.

(d) Are the new equilibria that appear in the pitchfork bifurcation attracting? (Suggestion: begin by looking at the bifurcation diagram.)