MA 732 Supplementary Problems

February 3, 2004

1. Define $F : C^0([a, b], R) \to R$ by $F(\phi) = \int_a^b (\phi(t))^2 \; dx$. Using the definition of derivative, prove that F is C^1, and $DF(\phi)\psi = \int_a^b 2\phi(t)\psi(t)\, dt$.

2. Let $Y_1, \ldots, Y_k$ by Banach spaces.

   (a) Show that $Y_1 \times \ldots \times Y_k$, with the norm $\|(y_1, \ldots, y_k)\| = \max_{1 \le i \le k} \|y_i\|_i$, is a Banach space. (In other words, show that the "norm" just defined is really a norm, and that $Y_1 \times \ldots \times Y_k$ with this norm is complete.)

   (b) Define $\Pi_i : Y_1 \times \ldots \times Y_k \to Y_i$ by $\Pi_i(y_1, \ldots, y_k) = y_i$. Show that Π_i is a bounded linear map with norm 1.

   (c) Let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Show that f is continuous if and only if each f_i is continuous. (It may simplify things to note that $f_i = \Pi_i \circ f$.)

   (d) Again let X be another Banach space, and let $f : X \to Y_1 \times \ldots \times Y_k$ be a map, $f(x) = (f_1(x), \ldots, f_k(x))$. Assume that each f_i is differentiable at a point $x \in X$. Show that f is differentiable at x, and

$$Df(x)h = (Df_1(x)h, \ldots, Df_k(x)h).$$