TRAPPING REGION FOR LOTKA-VOLTERRA WITH A LOGISTIC MODIFICATION

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Yesterday in class we showed the existence of a triangular trapping region for this equation. However, the argument was incomplete because I wrongly assumed that a certain parabola had to open upward. Here is a better argument, in which the fact that the parabola opens upward is a consequence of other assumptions.

System (sec. 4.3.2 in the text with different letters for the constants):
\[
\begin{align*}
\dot{x} &= x(1-y-\eta x), \\
\dot{y} &= \gamma y(x-1),
\end{align*}
\]
with \(\eta\) and \(\gamma\) positive. We assumed \(\eta < 1\) in order to have an interior equilibrium, but this is not needed to get a trapping region.

The desired trapping region is the triangle bounded on the left by \(x = 0\), below by \(y = 0\) (both these lines are invariant), and above by the line \(H(x, y) = xA + yB = 1\), \(A, B > 0\).

For a trapping region we need \(\nabla H \cdot (\dot{x}, \dot{y}) \leq 0\) when \(H = 1\) and \(0 \leq x \leq A\). We have
\[
\nabla H \cdot (\dot{x}, \dot{y}) = \frac{1}{A} x(1-y-\eta x) + \frac{1}{B} \gamma y(x-1).
\]

When \(H = 1\), \(y = B - Bx\). Substituting this expression into the above formula and simplifying, we find that when \(H = 1\), \(\nabla H \cdot (\dot{x}, \dot{y})\) is given by
\[
h(x) = \frac{1}{A} \left( \left( \frac{B}{A} - \eta - \gamma \right)x^2 + \left( 1 - B + \gamma A + \gamma \right)x - A\gamma \right).
\]
The graph of \(z = h(x)\) is a parabola.

For a trapping region we need
\[
h(x) \leq 0 \text{ for } 0 \leq x \leq A. \quad (0.1)
\]

Notice that

1. \(h(0) = -\gamma < 0\), i.e., the \(z\)-intercept of the parabola \(z = h(x)\), is negative.

We claim that if \(A\) and \(B\) are large enough that
\[
A \geq \frac{1}{\eta} \text{ and } B > (\eta + \gamma)A, \quad (0.2)
\]
then \((0.1)\) is satisfied.

Since \(B > (\eta + \gamma)A\),

2. the coefficient of \(x^2\) in \(h(x)\) is positive, i.e., the parabola \(z = h(x)\) opens upward.

In addition,
(3) \[ h(A) = \frac{1}{A} \left( \left( \frac{B}{A} - \eta - \gamma \right) A^2 + \left( 1 - B + \gamma A + \gamma \right) A - \gamma A \right) = -\eta A + 1 \leq 0. \]

The inequality follows from the assumption \( A \geq \frac{1}{\eta} \).

Now (1), (2), and (3) imply (0.1).