

# MA 532 Final Exam

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1. Consider the differential equation  $\dot{x} = x(x - 2)g(x)$ ,  $x \in R$ , with  $g(x) > 0$  for all  $x$ .
  - (a) Draw the phase portrait.
  - (b) The solution with  $x(0) = 4$  is defined on a maximal interval  $\alpha < t < \beta$ . Answer the following questions “yes” or “no”.
    - i. Is it possible that  $\alpha$  is finite?
    - ii. Is it possible that  $\alpha = -\infty$ ?
    - iii. Is it possible that  $\beta$  is finite?
    - iv. Is it possible that  $\beta = \infty$ ?
2. Let  $\dot{x} = f(t, x)$  be a  $C^1$  differential equation on  $R \times R^n$  with  $f(t + T, x) = f(t, x)$  for all  $(t, x)$ . Let  $x(t)$  be a solution. Let  $y(t) = x(t + T)$ .
  - (a) Show that  $y(t)$  is also a solution of  $\dot{x} = f(t, x)$ .
  - (b) Suppose that  $x(T) = x(0)$ . Show that  $y(t) = x(t)$  for all  $t$ .
3. Consider the differential equation  $\ddot{x} + 3x^2 = 1$ .
  - (a) Convert to a system by letting  $y = \dot{x}$ .
  - (b) Find the equilibria of the system.
  - (c) Draw the phase portrait of the system.
4. Consider the differential equation  $\ddot{x} + \epsilon\dot{x} + 3x^2 = 1$ ,  $\epsilon > 0$ . (We have added damping to Problem 3. Your work on Problem 3 will help you do this one.)
  - (a) Convert to a system by letting  $y = \dot{x}$ .
  - (b) Define an open set  $G$  in  $R^2$  such that if  $x_0 \in G$ , then the solution through  $x_0$  approaches the origin as  $t \rightarrow \infty$ .
  - (c) Use a Liapunov function suggested by your work on Problem 3 to *prove* that if  $x_0 \in G$ , then the solution through  $x_0$  approaches the origin as  $t \rightarrow \infty$ .

5. Consider the linear boundary value problem

$$\dot{x} = Ax + h(t), \quad 0 \leq t \leq 1,$$

$$x(0) = x(1),$$

where  $x \in \mathbb{R}^n$  and  $A$  is an  $n \times n$  matrix. Show that if 1 is not an eigenvalue of  $e^A$ , then this problem has a unique solution  $x(t)$ ,  $0 \leq t \leq 1$ . Hint: Write down the solution of  $\dot{x} = Ax + h(t)$  with  $x(0)$  given.

6. Consider the differential equation

$$\begin{aligned}\dot{x} &= 2xy \\ \dot{y} &= x^2 + y^2\end{aligned}$$

- Rewrite the system in polar coordinates.
- Divide by an appropriate power of  $r$ , and analyze the resulting system near the circle  $r = 0$ . As part of your analysis, you should find all equilibria on this circle and find the linearization at those equilibria.
- Use your analysis to draw the phase portrait of the original system near the origin.

7. Consider the  $2\pi$ -periodic differential equation

$$\dot{x} = x \sin^2 t - x^2.$$

Let  $P(\xi)$  be the Poincaré map.

- Show that  $x = 0$  is a solution of the differential equation, and that  $P'(0) > 1$ . (For the second part, use the linear variational equation.)
- Show that if  $x > 1$ , then  $\dot{x} < 0$ .
- Use parts (a) and (b) to show that there is a second  $2\pi$ -periodic solution.

8. Consider the differential equation

$$\begin{aligned}\dot{x} &= (x - 4)^2 + y^2 - 1 = x^2 + y^2 - 8x - 17, \\ \dot{y} &= x(x + y) = x^2 + xy.\end{aligned}$$

- Use the nullclines to draw the phase portrait in the finite plane. Are there any equilibria?
- Use the change of coordinates  $u = \frac{1}{x}$ ,  $v = \frac{y}{x}$  to show that if  $(x(t), y(t))$  is a solution, then  $\frac{y(t)}{x(t)} \rightarrow 1$  as  $t$  increases and  $\frac{y(t)}{x(t)} \rightarrow -1$  as  $t$  decreases. You may assume that there are no solutions with  $\frac{y(t)}{x(t)} \rightarrow \pm\infty$ . (A different change of coordinates would be needed to check this.)