

MA 532 Final Exam

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1. Consider the differential equation $\dot{x} = x(x - 2)g(x)$, $x \in R$, with $g(x) > 0$ for all x .
 - (a) Draw the phase portrait.
 - (b) The solution with $x(0) = 4$ is defined on a maximal interval $\alpha < t < \beta$. Answer the following questions “yes” or “no”.
 - i. Is it possible that α is finite?
 - ii. Is it possible that $\alpha = -\infty$?
 - iii. Is it possible that β is finite?
 - iv. Is it possible that $\beta = \infty$?
2. Let $\dot{x} = f(t, x)$ be a C^1 differential equation on $R \times R^n$ with $f(t + T, x) = f(t, x)$ for all (t, x) . Let $x(t)$ be a solution. Let $y(t) = x(t + T)$.
 - (a) Show that $y(t)$ is also a solution of $\dot{x} = f(t, x)$.
 - (b) Suppose that $x(T) = x(0)$. Show that $y(t) = x(t)$ for all t .
3. Consider the differential equation $\ddot{x} + 3x^2 = 1$.
 - (a) Convert to a system by letting $y = \dot{x}$.
 - (b) Find the equilibria of the system.
 - (c) Draw the phase portrait of the system.
4. Consider the differential equation $\ddot{x} + \epsilon\dot{x} + 3x^2 = 1$, $\epsilon > 0$. (We have added damping to Problem 3. Your work on Problem 3 will help you do this one.)
 - (a) Convert to a system by letting $y = \dot{x}$.
 - (b) Define an open set G in R^2 such that if $x_0 \in G$, then the solution through x_0 approaches the origin as $t \rightarrow \infty$.
 - (c) Use a Liapunov function suggested by your work on Problem 3 to *prove* that if $x_0 \in G$, then the solution through x_0 approaches the origin as $t \rightarrow \infty$.

5. Consider the linear boundary value problem

$$\dot{x} = Ax + h(t), \quad 0 \leq t \leq 1,$$

$$x(0) = x(1),$$

where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix. Show that if 1 is not an eigenvalue of e^A , then this problem has a unique solution $x(t)$, $0 \leq t \leq 1$. Hint: Write down the solution of $\dot{x} = Ax + h(t)$ with $x(0)$ given.

6. Consider the differential equation

$$\begin{aligned}\dot{x} &= 2xy \\ \dot{y} &= x^2 + y^2\end{aligned}$$

- Rewrite the system in polar coordinates.
- Divide by an appropriate power of r , and analyze the resulting system near the circle $r = 0$. As part of your analysis, you should find all equilibria on this circle and find the linearization at those equilibria.
- Use your analysis to draw the phase portrait of the original system near the origin.

7. Consider the 2π -periodic differential equation

$$\dot{x} = x \sin^2 t - x^2.$$

Let $P(\xi)$ be the Poincaré map.

- Show that $x = 0$ is a solution of the differential equation, and that $P'(0) > 1$. (For the second part, use the linear variational equation.)
- Show that if $x > 1$, then $\dot{x} < 0$.
- Use parts (a) and (b) to show that there is a second 2π -periodic solution.

8. Consider the differential equation

$$\begin{aligned}\dot{x} &= (x - 4)^2 + y^2 - 1 = x^2 + y^2 - 8x - 17, \\ \dot{y} &= x(x + y) = x^2 + xy.\end{aligned}$$

- Use the nullclines to draw the phase portrait in the finite plane. Are there any equilibria?
- Use the change of coordinates $u = \frac{1}{x}$, $v = \frac{y}{x}$ to show that if $(x(t), y(t))$ is a solution, then $\frac{y(t)}{x(t)} \rightarrow 1$ as t increases and $\frac{y(t)}{x(t)} \rightarrow -1$ as t decreases. You may assume that there are no solutions with $\frac{y(t)}{x(t)} \rightarrow \pm\infty$. (A different change of coordinates would be needed to check this.)