MA 532 Supplementary Problems 8 (corrected)

December 4, 2003

1. (Based on exercise 1.94.) Consider the differential equation

\[ \begin{align*}
\dot{x} &= x^2 + y^2 - 1, \\
\dot{y} &= 5(xy - 1).
\end{align*} \tag{1} \tag{2} \]

(a) Show that there are no equilibria.
(b) Use the method of nullclines to draw the phase portrait in the finite plane. (Draw the curves where \( \dot{x} = 0 \) and \( \dot{y} = 0 \). Determine the signs of \( \dot{x} \) and \( \dot{y} \) in the regions between these curves. Use this information to draw the phase portrait.)
(c) Show that there is a unique orbit \( \Gamma^+ \) for which \( \frac{y}{x} \to 0 \) as \( x \to \infty \), and a unique orbit \( \Gamma^- \) for which \( \frac{y}{x} \to 0 \) as \( x \to -\infty \). Suggestion: use the coordinates \( u = \frac{1}{x}, v = \frac{y}{x} \).
(d) On \( \Gamma^+ \), what does \( \frac{y}{x} \) approach as \( x \to -\infty \)? On \( \Gamma^- \), what does \( \frac{y}{x} \) approach as \( x \to \infty \)? Try to answer these questions by combining information from parts (b) and (c).

2. Suppose that \( b(t) \) is a 2\( \pi \)-periodic continuous function and let \( b_0 = \int_0^{2\pi} b(s) \, ds \). Show that all solutions of \( \dot{x} = b(t) \) are 2\( \pi \)-periodic if \( b_0 = 0 \); otherwise, they are all unbounded. Hint: Show that the Poincaré map is \( P(\xi) = \xi + b_0 \).

3. Fredholm’s Alternative. Suppose that \( a(t) \) and \( b(t) \) are 2\( \pi \)-periodic continuous functions, and let \( a_0 = \int_0^{2\pi} a(s) \, ds \). Show the following properties of the differential equation \( \dot{x} = a(t)x + b(t) \).

(a) If \( a_0 \neq 0 \), then there is a unique 2\( \pi \)-periodic orbit. It is asymptotically stable if \( a_0 < 0 \), and asymptotically unstable if \( a_0 > 0 \).
(b) Let \( c_0 = \int_0^{2\pi} \exp\{ \int_s^{2\pi} a(u) \, du \} b(s) \, ds \). If \( a_0 = 0 \), then every solution is 2\( \pi \)-periodic if and only if \( c_0 = 0 \).
(c) If \( a_0 = 0 \), then every solution is unbounded if \( c_0 \neq 0 \).

Hint: Show using the variation of constants formula that the Poincaré map is

\[ P(\xi) = e^{a_0} \xi + \int_0^{2\pi} \exp\{ \int_s^{2\pi} a(u) \, du \} b(s) \, ds, \]

and \( P(\xi) = \xi \) if and only if \( (1 - e^{a_0})\xi = c_0 \).
4. Show that the differential equation \( \dot{x} = -x^5 + c(t) \), where \( c(t) \) is a \( 2\pi \)-periodic continuous function, has a \( 2\pi \)-periodic solution. Show that any such solution is asymptotically stable. Explain why this implies that there is only one \( 2\pi \)-periodic solution.

5. Let \( c(t) \), \( d(t) \), and \( e(t) \) be \( 2\pi \)-periodic continuous functions. Show that the differential equation
   \[
   \dot{x} = -x^3 + c(t)x^2 + d(t)x + e(t)
   \]
has at least one \( 2\pi \)-periodic solution. Show that if this solution is unstable, then there must be another \( 2\pi \)-periodic solution. Hint: Show that \( \dot{x} \) is negative if \( x \) is sufficiently positive, and positive if \( x \) is sufficiently negative.

6. Riccati Equation. Suppose that \( a(t) \) and \( b(t) \) are \( 2\pi \)-periodic continuous functions. Prove that the Riccati equation
   \[
   \dot{x} = b(t) + a(t)x - x^2
   \]
has at most two \( 2\pi \)-periodic solutions. Hint: Suppose that \( \phi(t) \) is a \( 2\pi \)-periodic solution. If \( x(t) \) is another solution, let \( y(t) = x(t) - \phi(t) \). Show that
   \[
   \dot{y} = c(t)y - y^2,
   \]
where \( c(t) = a(t) - 2\phi(t) \). Then let \( w(t) = \frac{1}{y(t)} \). Show that
   \[
   \dot{w} = -c(t)w + 1
   \]
Use the Fredholm Alternative to discuss separately the cases \( \int_0^{2\pi} c(t) \, dt \neq 0 \) and \( \int_0^{2\pi} c(t) \, dt = 0 \).

7. Show that \( x = \sin t \) is a \( 2\pi \)-periodic solution of the differential equation
   \[
   \dot{x} = -x^3 + 2x + \sin^3 t - 2 \sin t + \cos t.
   \]
Show that this solution is unstable with \( P'(0) = e^{2\pi i} \). How many more \( 2\pi \)-periodic solutions can you guarantee?

8. Suppose that \( a(t) \) is \( 2\pi \)-periodic with \( 0 < a(t) < 1 \) for all \( t \). Show that the differential equation \( \dot{x} = x(x - a(t))(x - 1) \) has at least three \( 2\pi \)-periodic solutions. Hint: Show that \( x(t) = 0 \) and \( x(t) = 1 \) are asymptotically stable, and explain why this implies that there is a \( 2\pi \)-periodic solution between them.