

MA 532 Supplementary Problems 6

October 24, 2003

This homework is an expanded version of Problem 2.30 in the text.

- Recall that $(A + B)^\top = A^\top + B^\top$, $(cA)^\top = cA^\top$, and $(A_1A_2 \dots A_k)^\top = A_k^\top \dots A_2^\top A_1^\top$.
 - Let A be an $n \times n$ matrix. Let $p(x)$ be a polynomial, $p(x) = \sum_{i=0}^k a_i x^i$. Show that $(p(A))^\top = p(A^\top)$.
 - Explain why (a) implies that $(e^{tA})^\top = e^{tA^\top}$.
- Let A be an $n \times n$ matrix. Recall that A is *skew-symmetric* if $A^\top = -A$, and A is *orthogonal* if $A^\top A = I$.
 - Show: If S is skew-symmetric, then S and S^\top commute.
 - Using (a) and 1(b), show: If S is skew-symmetric, then e^{tS} is orthogonal.
- Let S be a 3×3 skew-symmetric matrix whose entries are not all 0.
 - Show that the eigenvalues of S are 0 and $\pm\beta i$, $\beta > 0$. Hint: $S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$.
 - Let $x(t)$ be a solution of $\dot{x} = Sx$. Show that $x(t + \frac{2\pi}{\beta}) = x(t)$.
 - Let v be an eigenvector of S for the eigenvalue 0. Try to explain intuitively why multiplication by e^{tS} is just rotation about the direction v , through the angle βt .
- Let v be a vector in R^3 with $v \neq 0$. Consider the differential equation $\dot{x} = v \times x$.
 - Find a 3×3 matrix S such that $\dot{x} = Sx$. Show that S is skew-symmetric.
 - By 3(a), the eigenvalues of S are 0 and $\pm\beta i$. What is the eigenvector for the eigenvalue 0? How is β related to the length of v ?