MA 532 Supplementary Problems 6

October 24, 2003

This homework is an expanded version of Problem 2.30 in the text.

1. Recall that $(A + B)^T = A^T + B^T$, $(cA)^T = cA^T$, and $(A_1 A_2 \ldots A_k)^T = A_k^T \ldots A_2^T A_1^T$.
   
   (a) Let $A$ be an $n \times n$ matrix. Let $p(x)$ be a polynomial, $p(x) = \Sigma_{i=0}^{k} a_i x^i$. Show that $(p(A))^T = p(A^T)$.
   
   (b) Explain why (a) implies that $(e^{tA})^T = e^{tA^T}$.

2. Let $A$ be an $n \times n$ matrix. Recall that $A$ is skew-symmetric if $A^T = -A$, and $A$ is orthogonal if $A^T A = I$.
   
   (a) Show: If $S$ is skew-symmetric, then $S$ and $S^T$ commute.
   
   (b) Using (a) and 1(b), show: If $S$ is skew-symmetric, then $e^{tS}$ is orthogonal.

3. Let $S$ be a $3 \times 3$ skew-symmetric matrix whose entries are not all 0.
   
   (a) Show that the eigenvalues of $S$ are 0 and $\pm \beta i$, $\beta > 0$. Hint: $S = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$.
   
   (b) Let $x(t)$ be a solution of $\dot{x} = Sx$. Show that $x(t + \frac{2\pi}{\beta}) = x(t)$.
   
   (c) Let $v$ be an eigenvector of $S$ for the eigenvalue 0. Try to explain intuitively why multiplication by $e^{tS}$ is just rotation about the direction $v$, through the angle $\beta t$.

4. Let $v$ be a vector in $R^3$ with $v \neq 0$. Consider the differential equation $\dot{x} = v \times x$.
   
   (a) Find a $3 \times 3$ matrix $S$ such that $\dot{x} = Sx$. Show that $S$ is skew-symmetric.
   
   (b) By 3(a), the eigenvalues of $S$ are 0 and $\pm \beta i$. What is the eigenvector for the eigenvalue 0? How is $\beta$ related to the length of $v$?