MA 532 Supplementary Problems 3

September 12, 2003

1. Consider the differential equation $\ddot{x} - x + x^3 = 0$.

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Find the equilibria of the system.

   (c) Draw the phase portrait of the system.

2. Consider the differential equation $\ddot{x} + \epsilon\dot{x} - x + x^3 = 0$, $\epsilon > 0$. (We have added damping to Problem 1. Your work on Problem 1 will help you do this one.)

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Use a Lyapunov function suggested by your work on Problem 1, together with Lasalle's Invariance Principle, to prove that every solution approaches one of the equilibria as $t \to \infty$.

3. Consider the differential equation $\ddot{x} + \dot{x} + x - x^3 = 0$. (We have added damping to Exercise 1.36 (3) on p. 27. Your work on that problem will help you do this one.)

   (a) Convert to a system by letting $y = \dot{x}$.

   (b) Define an open set G in R^2 such that if $p \in G$, then the solution through p approaches the origin as $t \to \infty$.

   (c) Use a Lyapunov function suggested by your work on Exercise 1.36 (3), together with Lasalle's Invariance Principle, to *prove* that if $p \in G$, then the solution through p approaches the origin as $t \to \infty$.

4. Let $\dot{x} = f(x)$ be a C^1 differential equation on R^n, and let $p \in R^n$. Let $\Gamma = \{\varphi(t,p) : t \geq 0\}$, the positive semi-orbit through p. We use $\mathrm{cl}(A)$ to denote the closure of a set A.

   (a) Prove: $\mathrm{cl}(\Gamma) = \Gamma \cup \omega(p)$.

   (b) Prove: If $\omega(p)$ is empty, then Γ is closed.

   (c) Prove: If Γ is compact, then $\Gamma = \omega(p)$.