MA 532 Supplementary Problems 2

August 29, 2003

1. (Based on Exercise 1.6, p. 4.) Consider the differential equation $\dot{x} = -\sqrt{|x|}$. Find all solutions that satisfy $x(0) = 0$. (This is similar to the initial value problem $\dot{x} = x^{\frac{2}{3}}$ discussed in class. Hint: when $x < 0$, $|x| = -x$.)

2. (Based on Exercise 1.9, p. 10.) Suppose $F : R \to R$ is a C^1 positive periodic function with period $p > 0$:
$$F(x + p) = F(x) \text{ for all } x. \tag{1}$$
Consider the differential equation
$$\frac{dx}{dt} = F(x). \tag{2}$$
Let $x(t)$ be the solution with $x(0) = 0$.

   (a) Explain in about three words why $x(t)$ is an increasing function of t.

   (b) Since $F(x)$ is a continuous positive periodic function, there exists a positive number K such that $F(x) \le K$ for all x. Show that for any $t_1 < t_2$, $0 < x(t_2) - x(t_1) \le K(t_2 - t_1)$. Hint: By the Fundamental Theorem of Calculus, $x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t)\, dt$.

   (c) Use (b) and Theorem 1.4 to explain why $x(t)$ is defined for $-\infty < t < \infty$.

   (d) Let T be a positive number such that
$$x(T) = p. \tag{3}$$
(Why does such a number exist?) Let $y(t) = x(t + T) - x(T)$. By differentiating this expression, show that $\dot{y}(t) = F(y(t))$. You will need to use (1), (2), and (3).

   (e) Explain why uniqueness of solutions implies that $y(t) = x(t)$ for all t. Conclude that $x(t + T) = x(t) + p$ for all t.

   (f) Explain why
$$\int_0^p \frac{1}{F(x)}\, dx = T.$$
Suggestion: $x(t)$ is an increasing function of t, so it has an inverse function $t(x)$, and
$$\frac{dt}{dx} = \frac{1}{F(x)}. \tag{4}$$

(We make use of this differential equation when we solve (2) by separation of variables.) Both sides of (4) are functions of x. Integrate both sides from $x = 0$ to $x = p$.

3. (Based on Exercise 1.10, p. 13.)

   (a) For each nonzero integer p, construct the flow $\phi(t, x_0)$ of $\dot{x} = x^p$. (You will need to treat the case $p = 1$ separately.)

   (b) For each flow that you construct, verify that $\phi(t + s, x_0) = \phi(t, \phi(s, x_0))$.

4. The differential equation $\ddot{u} + \omega^2 u = 0$, rewritten as a system with $x = u$ and $y = \dot{u}$, becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega^2 x. \end{aligned}$$

   (a) Find the flow $\phi(t, (x_0, y_0))$. Hint: The general solution of $\ddot{u} + \omega^2 u = 0$ is $u = c_1 \cos \omega t + c_2 \sin \omega t$.

   (b) Verify that $\phi(t + s, (x_0, y_0)) = \phi(t, \phi(s, (x_0, y_0)))$.