1. (Based on Exercise 1.6, p. 4.) Consider the differential equation \( \dot{x} = -\sqrt{|x|} \). Find all solutions that satisfy \( x(0) = 0 \). (This is similar to the initial value problem \( \dot{x} = x^{\frac{2}{3}} \) discussed in class. Hint: when \( x < 0 \), \( |x| = -x \).)

2. (Based on Exercise 1.9, p. 10.) Suppose \( F : R \to R \) is a \( C^1 \) positive periodic function with period \( p > 0 \):

\[
F(x + p) = F(x) \quad \text{for all } x. \tag{1}
\]

Consider the differential equation

\[
\frac{dx}{dt} = F(x). \tag{2}
\]

Let \( x(t) \) be the solution with \( x(0) = 0 \).

(a) Explain in about three words why \( x(t) \) is an increasing function of \( t \).

(b) Since \( F(x) \) is a continuous positive periodic function, there exists a positive number \( K \) such that \( F(x) \leq K \) for all \( x \). Show that for any \( t_1 < t_2 \), \( 0 < x(t_2) - x(t_1) \leq K(t_2 - t_1) \). Hint: By the Fundamental Theorem of Calculus, \( x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t) \, dt \).

(c) Use (b) and Theorem 1.4 to explain why \( x(t) \) is defined for \(-\infty < t < \infty\).

(d) Let \( T \) be a positive number such that \( x(T) = p \). (Why does such a number exist?) Let \( y(t) = x(t + T) - x(T) \). By differentiating this expression, show that \( \dot{y}(t) = F(y(t)) \). You will need to use (1), (2), and (3).

(e) Explain why uniqueness of solutions implies that \( y(t) = x(t) \) for all \( t \). Conclude that \( x(t + T) = x(t) + p \) for all \( t \).

(f) Explain why

\[
\int_{0}^{p} \frac{1}{F(x)} \, dx = T.
\]

Suggestion: \( x(t) \) is an increasing function of \( t \), so it has an inverse function \( t(x) \), and

\[
\frac{dt}{dx} = \frac{1}{F(x)}. \tag{4}
\]
(We make use of this differential equation when we solve (2) by separation of variables.) Both sides of (4) are functions of $x$. Integrate both sides from $x = 0$ to $x = p$.

3. (Based on Exercise 1.10, p. 13.)

(a) For each nonzero integer $p$, construct the flow $\phi(t, x_0)$ of $\dot{x} = x^p$. (You will need to treat the case $p = 1$ separately.)

(b) For each flow that you construct, verify that $\phi(t + s, x_0) = \phi(t, \phi(s, x_0))$.

4. The differential equation $\ddot{u} + \omega^2 u = 0$, rewritten as a system with $x = u$ and $y = \dot{u}$, becomes

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -\omega^2 x.
\end{align*}
\]

(a) Find the flow $\phi(t, (x_0, y_0))$. Hint: The general solution of $\ddot{u} + \omega^2 u = 0$ is $u = c_1 \cos \omega t + c_2 \sin \omega t$.

(b) Verify that $\phi(t + s, (x_0, y_0)) = \phi(t, \phi(s, (x_0, y_0)))$. 