

MA 532 Supplementary Problems 1

August 20, 2003

1. Sketch the phase portrait, the direction field, and some typical solutions.

(a) $\dot{x} = x - x^3$

(b) $\dot{x} = 1 + x^2$

(c) $\dot{x} = x^2 - x^3$

(d) $\dot{x} = e^x \sin x$

(e) $\dot{x} = 1 - 2 \cos x$

2. The velocity $v(t)$ of a falling skydiver is governed by the differential equation

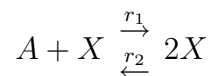
$$m\dot{v} = mg - kv^2,$$

where m is the mass of the skydiver, g is the acceleration due to gravity, and k is a constant related to air resistance. The constants m , g , and k are positive.

(a) Sketch the phase portrait.

(b) What is the skydiver's terminal velocity?

3. Consider the model chemical reaction



in which one molecule of X combines with one molecule of A to form two molecules of X . This means that the chemical X stimulates its own production, a process called *autocatalysis*. This positive feedback process is eventually limited by a “back reaction” in which $2X$ returns to $A+X$. According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations of A and X by a and x respectively. Assume that there is an enormous surplus of chemical A , so that its concentration a can be regarded as constant. Then the differential equation for this reaction is

$$\dot{x} = r_1 a x - r_2 x^2$$

where r_1 and r_2 are positive *rate constants*.

- (a) Sketch the phase portrait.
- (b) What value does the concentration of X approach?

4. Sketch the bifurcation diagram.

- (a) $\dot{x} = \mu x - x^3$. Why is this called a pitchfork bifurcation?
- (b) $\dot{x} = \mu^2 - x^2$

5. Consider a bead on circular hoop that is hanging from the ceiling and spinning. A *rough* model, assuming considerable damping, is given by the differential equation

$$\dot{\theta} = (\mu \cos \theta - 1) \sin \theta$$

where θ is the angular distance of the bead from the bottom of the hoop and μ is related to how fast the hoop is spinning. For simplicity we only consider $-\pi < \theta < \pi$.

- (a) Draw the phase portrait in the region $-\pi < \theta < \pi$ for $\mu = \frac{1}{2}$. What does the bead do?
- (b) Draw the phase portrait in the region $-\pi < \theta < \pi$ for $\mu = 2$. What does the bead do?
- (c) Draw the bifurcation diagram in the region $-\pi < \theta < \pi$ and $\mu > 0$.
- (d) If we made our θ -region a little bigger, the phase portrait would show equilibria at $\theta = \pm\pi$. What is the physical meaning of these equilibria?

6. Sketch the phase portrait.

- (a) $\ddot{x} + x - x^3 = 0$
- (b) $\ddot{x} + x + x^3 = 0$
- (c) $\ddot{x} - x - x^3 = 0$
- (d) $\ddot{x} - x + x^3 = 0$
- (e) $\ddot{x} + x - x^2 = 0$

7. Another model for a bead on a rotating hoop, assuming *no* damping, is

$$\ddot{\theta} = (\mu \cos \theta - 1) \sin \theta.$$

Draw the phase portraits in the region $-\frac{3}{2}\pi < \theta < \frac{3}{2}\pi$, $-\infty < \dot{\theta} < \infty$ for $\mu = \frac{1}{2}$ and $\mu = 2$.