# MA 532 Supplementary Problems 1

August 20, 2003

1. Sketch the phase portrait, the direction field, and some typical solutions.

   (a) $\dot{x} = x - x^3$

   (b) $\dot{x} = 1 + x^2$

   (c) $\dot{x} = x^2 - x^3$

   (d) $\dot{x} = e^x \sin x$

   (e) $\dot{x} = 1 - 2\cos x$

2. The velocity $v(t)$ of a falling skydiver is governed by the differential equation

$$m\dot{v} = mg - kv^2,$$

   where $m$ is the mass of the skydiver, $g$ is the acceleration due to gravity, and $k$ is a constant related to air resistance. The constants $m$, $g$, and $k$ are positive.

   (a) Sketch the phase portrait.

   (b) What is the skydiver's terminal velocity?

3. Consider the model chemical reaction

$$A + X \underset{r_2}{\overset{r_1}{\rightleftarrows}} 2X$$

   in which one molecule of $X$ combines with one molecule of $A$ to form two molecules of $X$. This means that the chemical $X$ stimulates its own production, a process called *autocatalysis*. This positive feedback process is eventually limited by a "back reaction" in which $2X$ returns to $A+X$. According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations of $A$ and $X$ by $a$ and $x$ respectively. Assume that there is an enormous surplus of chemical $A$, so that its concentration $a$ can be regarded as constant. Then the differential equation for this reaction is

$$\dot{x} = r_1 a x - r_2 x^2$$

   where $r_1$ and $r_2$ are positive *rate constants*.

(a) Sketch the phase portrait.

(b) What value does the concentration of $X$ approach?

4. Sketch the bifurcation diagram.

   (a) $\dot{x} = \mu x - x^3$. Why is this called a pitchfork bifurcation?

   (b) $\dot{x} = \mu^2 - x^2$

5. Consider a bead on circular hoop that is hanging from the ceiling and spinning. A *rough* model, assuming considerable damping, is given by the differential equation

$$\dot{\theta} = (\mu \cos\theta - 1)\sin\theta$$

where $\theta$ is the angular distance of the bead from the bottom of the hoop and $\mu$ is related to how fast the hoop is spinning. For simplicity we only consider $-\pi < \theta < \pi$.

   (a) Draw the phase portrait in the region $-\pi < \theta < \pi$ for $\mu = \frac{1}{2}$. What does the bead do?

   (b) Draw the phase portrait in the region $-\pi < \theta < \pi$ for $\mu = 2$. What does the bead do?

   (c) Draw the bifurcation diagram in the region $-\pi < \theta < \pi$ and $\mu > 0$.

   (d) If we made our $\theta$-region a little bigger, the phase portrait would show equilibria at $\theta = \pm\pi$. What is the physical meaning of these equilibria?

6. Sketch the phase portrait.

   (a) $\ddot{x} + x - x^3 = 0$

   (b) $\ddot{x} + x + x^3 = 0$

   (c) $\ddot{x} - x - x^3 = 0$

   (d) $\ddot{x} - x + x^3 = 0$

   (e) $\ddot{x} + x - x^2 = 0$

7. Another model for a bead on a rotating hoop, assuming *no* damping, is

$$\ddot{\theta} = (\mu \cos\theta - 1)\sin\theta.$$

Draw the phase portraits in the region $-\frac{3}{2}\pi < \theta < \frac{3}{2}\pi$, $-\infty < \dot{\theta} < \infty$ for $\mu = \frac{1}{2}$ and $\mu = 2$.