MA 493 Homework 3

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This homework is based on problem 3.12 in the text (the "Rotten Kid Theorem"). I've tried to make it a little more precise.

The problem concerns a rotten son and a loving mother. The son works in a family business that provides income to both himself and his mother. The son decides how hard to work, which we represent by a number a. This decision affects his mother's annual income $y(a)$ and his own annual income $z(a)$. The mother then observes her own income y and her son's income z, and decides on an amount t between 0 and y to give to her son. Thus the amount the mother has left to spend is $y(a) - t$, and the amount the son has to spend is $z(a) + t$.

The mother has a utility function u that relates the amount she has to spend to her happiness. Similarly, the son has a utility function v that relates the amount he has to spend to his happiness. Both u and v have positive first derivative and negative second derivative, as usual.

The son's strategy is the number a and the mother's strategy is the number t. These numbers determine the son's payoff $\Pi_1(a, t)$ and the mother's payoff $\Pi_2(a, t)$.

The son is rotten: he only cares about himself. Thus

$$\Pi_1(a, t) = v\big(z(a) + t\big).$$

The mother is not rotten: she cares about both herself and her son. Thus

$$\Pi_2(a, t) = u\big(y(a) - t\big) + \alpha v\big(z(a) + t\big),$$

where the positive number α is her coefficient of altruism.

Since the son goes first, he decides what to do by rollback.

1. The son thinks: If I choose a, Mom will choose $t(a)$ to maximize her payoff. Show that if $0 < t(a) < y$, then $t = t(a)$ must satisfy the equation
$$-u'\big(y(a) - t\big) + \alpha v'\big(z(a) + t\big) = 0. \tag{1}$$
(Just set $\frac{\partial \Pi_2}{\partial t} = 0$).

2. Show that if $t = t(a)$ satisfies (1), then, for fixed a, $\Pi_2(a, t)$ is maximum at $t = t(a)$. (Look at $\frac{\partial^2 \Pi_2}{\partial t^2}$.)

3. Using rollback reasoning, the son chooses $a = a^*$ to maximize a function we shall call f:
$$f(a) = \Pi_1\big(a, t(a)\big) = v\big(z(a) + t(a)\big).$$
Show that
$$z'(a^*) + t'(a^*) = 0. \tag{2}$$
(Just set $f'(a) = 0$.)

4. Show that
$$y'(a^*) - t'(a^*) = 0. \tag{3}$$
Hint: Let $g(a) = -u'\big(y(a) - t(a)\big) + \alpha v'\big(z(a) + t(a)\big)$. From problem 1, $g(a) \equiv 0$. Therefore $g'(a) \equiv 0$. Calculate $g'(a)$, set it equal to 0, and set a equal to a^*.

5. Equations (2) and (3) imply that $y'(a^*) + z'(a^*) = 0$. Explain why a son who was trying to maximize total family income $y + z$ would choose the same level of effort a^*.

Thus Becker concludes that maybe the son is not so rotten after all!