1. In a certain town, there are two stores, a grocery store and a gas station. The grocery store charges $p_1$ dollars per pound for food, and the gas station charges $p_2$ dollars per gallon for gas. The grocery store sells $q_1$ pounds of food per week, and the gas station sells $q_2$ gallons of gas per week. The quantities $q_1$ and $q_2$ are related to the prices $p_1$ and $p_2$ as follows:

\[
q_1 = 10 - 2p_1 - p_2, \\
q_2 = 10 - p_1 - 2p_2.
\]

Thus, if the price of food or gas rises, less of both is sold.

Let $\pi_1$ be the revenue of the grocery store and $\pi_2$ the revenue of the gas station. Both depend on the two stores’ choices of $p_1$ and $p_2$:

\[
\pi_1(p_1, p_2) = q_1p_1 = (10 - 2p_1 - p_2)p_1 = 10p_1 - 2p_1^2 - p_1p_2, \\
\pi_2(p_1, p_2) = q_2p_2 = (10 - p_1 - 2p_2)p_2 = 10p_2 - p_1p_2 - 2p_2^2.
\]

We interpret this as a game with two players, the grocery store (player 1) and the gas station (player 2). The payoff to each player is its revenue.

We shall allow $p_1$ and $p_2$ to be any real numbers, even negative numbers, and even numbers that produce negative values for $q_1$ and $q_2$.

Suppose the grocery store chooses its price $p_1$ first, and then the gas station, knowing $p_1$, chooses its price $p_2$. If the grocery store uses backward induction to choose $p_1$, what price will it choose? What will be the corresponding $p_2$, and what will be the revenue of each store?

Partial answer to help keep you on the right track: $p_1 = 2\frac{1}{7}$.

2. We will now change the problem a little. First we change the formulas for $q_1$ and $q_2$ to the following more reasonable formulas, which say that
when the prices get too high, the quantities sold become 0, not negative numbers:

\[
q_1 = \begin{cases} 
10 - 2p_1 - p_2 & \text{if } 2p_1 + p_2 < 10 \\
0 & \text{if } 2p_1 + p_2 \geq 10
\end{cases}
\]

\[
q_2 = \begin{cases} 
10 - p_1 - 2p_2 & \text{if } p_1 + 2p_2 < 10 \\
0 & \text{if } p_1 + 2p_2 \geq 10
\end{cases}
\]

Next we make some reasonable restrictions on the prices. First we assume

\[ p_1 \geq 0 \text{ and } p_2 \geq 0. \]

Next we note that if \( p_1 \geq 0 \) and \( p_2 > 5 \), then \( q_2 \) becomes 0. The gas station wouldn’t want this, so we assume

\[ p_2 \leq 5. \]

Finally we note that if \( p_2 \geq 0 \) and \( p_1 > 5 \), then \( q_1 \) becomes 0. The grocery store wouldn’t want this, so we assume

\[ p_1 \leq 5. \]

The formulas for \( \pi_1 \) and \( \pi_2 \) are now a little different, because the formulas for \( q_1 \) and \( q_2 \) have changed. The payoffs \( \pi_1 \) and \( \pi_2 \) are now only defined for \( 0 \leq p_1 \leq 5 \) and \( 0 \leq p_2 \leq 5 \).

We still assume that the grocery store chooses its price \( p_1 \) first, and then the gas station, knowing \( p_1 \), chooses its price \( p_2 \).

(a) Write down formulas for \( \pi_1(p_1, p_2) \) and \( \pi_2(p_1, p_2) \), which are defined for \( 0 \leq p_1 \leq 5 \) and \( 0 \leq p_2 \leq 5 \). Like the formulas for \( q_1 \) and \( q_2 \), they will have two parts.

(b) For a fixed \( p_1 \) between 0 and 5, graph the function \( \pi_2(p_1, p_2) \), which is a function of \( p_2 \) defined for \( 0 \leq p_2 \leq 5 \). Answer: should be partly an upside down parabola and partly a horizontal line. Make sure you clearly indicate the point at which the graph changes from one to the other.

(c) By referring to the graph you just drew and using calculus, find the gas station’s best response function \( p_2 = b(p_1) \), which should be defined for \( 0 \leq p_1 \leq 5 \). Answer: \( b(p_1) = (10 - p_1)/4 \).

(d) You are now ready to find \( p_1 \) by backward induction. From your formula for \( \pi_1 \) you should be able to see that

\[
\pi_1(p_1, b(p_1)) = \begin{cases} 
(10 - 2p_1 - b(p_1))p_1 & \text{if } 2p_1 + b(p_1) < 10, \\
0 & \text{if } 2p_1 + b(p_1) \geq 10,
\end{cases}
\]
Use the formula for $b(p_1)$ from part (c) to show that $2p_1 + b(p_1) < 10$ if $0 \leq p_1 < \frac{42}{7}$, and $2p_1 + b(p_1) \geq 10$ if $\frac{42}{7} \leq p_1 \leq 5$.

(e) Graph the function $\pi_1(p_1, b(p_1))$, $0 \leq p_1 \leq 5$. (Again, it is partly an upside down parabola and partly a horizontal line.)

(f) By referring to the graph you just drew and using calculus, find where $\pi_1(p_1, b(p_1))$ is maximum.