MA 440 Homework 3

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This problem is related to Sec. 1.11 in the online notes, which should help you if you get stuck.

A rotten son manages a family business. The amount of effort the son puts into the business affects both his income and his mother's. The son, being rotten, cares only about his own income, not his mother's. To make matters worse, Mother dearly loves her son. If the son's income is low, Mother will give part of her own income to her son so that he will not suffer. (Mother has other sources of income besides this family business.) In this situation, can the son be expected to do what is best for the family?

We denote the mother's annual income by y and the son's by z. The amount of effort that the son devotes to the family business is denoted by a. His choice of a will affect both his income and his mother's, so we regard both y and z as functions of a: $y = y(a)$ and $z = z(a)$.

After mother observes a, and hence observes her own income $y(a)$ and her son's income $z(a)$, she chooses an amount t, $0 \leq t \leq y(a)$, to give to her son.

The mother and son have personal utility functions u and v respectively. Each is a function of the amount they have to spend.

The son chooses his effort a to maximize his own utility v, without regard for his mother's utility u. Mother, however, chooses the amount t to transfer to her son to maximize $u(y - t) + \alpha v(z + t)$, where α is her coefficient of altruism. Thus the payoff functions for this game are

$$\Pi_1(a, t) = v(z(a) + t),$$
$$\Pi_2(a, t) = u(y(a) - t) + \alpha v(z(a) + t).$$

Since the son chooses first, he can use backward induction to decide how much effort to put into the family business. In other words, he can take into account that even if he doesn't put in much effort, and so doesn't produce much income for either himself or his mother, his mother will help him out.

Assumptions:

1. The functions u and v have positive first derivative and negative second derivative.

2. The son's level of effort is chosen from an interval $I = [a_1, a_2]$.

3. Let $T(a) = y(a) + z(a)$ denote total family income. Then $T'(a) = 0$ at a unique point $a^\sharp$, $a_1 < a^\sharp < a_2$, and $T(a)$ attains its maximum value at this point. This assumption expresses the idea that if the son works too hard, he will do more harm than good. As they say in the software industry, if you stay at work too late, you're just adding bugs.

Let's first figure out the amount $t = b(a)$ that Mother will transfer to her son if she observes that his level of effort is a.

(a) Explain briefly why Mother will want to maximize the function $\Pi_2(a, t)$ with a fixed and $0 \leq t \leq y(a)$.

(b) Find a formula for $\frac{\partial \Pi_2}{\partial t}(a, t)$.

(c) Give assumptions that guarantee that $\frac{\partial \Pi_2}{\partial t}(a, 0) > 0$ and $\frac{\partial \Pi_2}{\partial t}(a, y(a)) < 0$. Discuss briefly whether these assumptions are reasonable.

(d) Calculate $\frac{\partial^2 \Pi_2}{\partial t^2}(a, t)$ and show that it is always negative.

Parts (c) and (d) imply that there is a single value of t where $\Pi_2(a, t)$, a fixed, attains its maximum; moreover, $0 < t < y(a)$, so $\frac{\partial \Pi_2}{\partial t}(a, t) = 0$. We denote this value of t by $t = b(a)$. This is Mother's strategy, the amount Mother will give to her son if his level of effort in the family business is a

The son now chooses his level of effort $a = a^*$ to maximize the function $\Pi_1(a, b(a))$, which we shall denote $V(a)$:

$$V(a) = \Pi_1(a, b(a)) = v(z(a) + b(a)).$$

Mother then contributes $t^* = b(a^*)$.

(e) Suppose $a_1 < a^* < a_2$ (the usual case). Then $V'(a^*) = 0$. Show that

$$z'(a^*) + b'(a^*) = 0. \tag{1}$$

(f) Explain why for every a, $\frac{\partial \Pi_2}{\partial t}(a, b(a)) = 0$. Write out this equation in detail, differentiate with respect to a, and set $a = a^*$ to show that

$$y'(a^*) - b'(a^*) = 0. \qquad (2)$$

(You will need to use (1).)

(g) Add (2) and (1) to obtain

$$y'(a^*) + z'(a^*) = 0.$$

Explain why this equation implies that $a^* = a^\sharp$, the level of effort that maximizes total family income.

Thus, if the son had not been rotten, and instead had been trying to maximize total family income $y(a) + z(a)$, he would have chosen the same level of effort a^*.