1. A city council consists of three members: members 1, 2, and 3. The council must name a new city administrator. There are three candidates: $A$, $B$, and $C$. The council members’ preferences are:

<table>
<thead>
<tr>
<th>Council member</th>
<th>first choice</th>
<th>second choice</th>
<th>third choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A$</td>
<td>$B$</td>
<td>$C$</td>
</tr>
<tr>
<td>2</td>
<td>$B$</td>
<td>$C$</td>
<td>$A$</td>
</tr>
<tr>
<td>3</td>
<td>$C$</td>
<td>$A$</td>
<td>$B$</td>
</tr>
</tbody>
</table>

Each council member votes for one candidate. If there is a tie (i.e., if each candidate gets one vote), council member 3, who is chair of the city council, breaks the tie. In the event of a tie, council member 3 will of course choose her favorite candidate, $C$, even if for some reason she did not vote for $C$ at first.

The payoffs to a member are 2 if the member’s first choice is selected, 1 if the member’s second choice is selected, and 0 if the member’s third choice is selected.

There is no way a member can benefit by voting for her third choice. Therefore each member has two options: vote for her first choice or vote for her second choice.

The payoffs are given by the following table.
### Member 3 votes for C

<table>
<thead>
<tr>
<th></th>
<th>Member 2 votes for</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Member 1 votes for A</td>
<td>(0,1,2)</td>
<td>(0,1,2)</td>
</tr>
<tr>
<td></td>
<td>(1,2,0)</td>
<td>(0,1,2)</td>
</tr>
</tbody>
</table>

### Member 3 votes for A

<table>
<thead>
<tr>
<th></th>
<th>Member 2 votes for</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>Member 1 votes for A</td>
<td>(2,0,1)</td>
<td>(2,0,1)</td>
</tr>
<tr>
<td></td>
<td>(1,2,0)</td>
<td>(0,1,2)</td>
</tr>
</tbody>
</table>

(a) Explain the entries in the second row of the second matrix. (In other words, explain why the payoffs in the second row of the second matrix are (1,2,0) and (0,1,2)).

(b) Justify the following statement: Member 3’s strategy “vote for C” weakly dominates her strategy “vote for A.”

(c) Use iterated elimination of weakly dominated strategies to find a Nash equilibrium. Describe the order in which you eliminate strategies.

(d) Find all the other pure strategy Nash equilibria in this game.
2. Two firms produce all the world’s widgets. Let
   - \( s \) = number of widgets made by firm 1.
   - \( t \) = number of widgets made by firm 2.

The price \( p \) of widgets depends on \( s + t \):

\[
p = \begin{cases} 
1000 - 2(s + t) & \text{if } s + t < 500, \\
0 & \text{if } s + t \geq 500.
\end{cases}
\]

At this price, all the widgets that are produced are sold. Notice that if no widgets are produced, the price is 1000. As the number of widgets produced rises, the price falls. Once 500 or more widgets are produced, the price is 0.

The cost of producing a widget is 20. Therefore, if 490 or more widgets are produced, the cost of producing a widget is more than the price it can be sold for.

Each firm wants to maximize its profit. Since profit is revenue minus cost, the firms’ profits are given by

\[
\Pi_1(s, t) = ps - 20s, \\
\Pi_2(s, t) = pt - 20t,
\]

where \( p \) is given by the above formula.

(a) Find a Nash equilibrium with \( s + t < 490 \).

(b) Are there any Nash equilibria with \( s + t \geq 490 \)? Explain.

(c) Suppose firm 1 goes first and chooses a level of production \( s \) with \( 0 < s < 490 \). What level of production \( t \) should firm 2 choose to maximize its profit?

(d) Suppose firm 1 goes first and chooses its level of production \( s \) using backward induction. Use your answer to part (c) to decide what level of production it should choose.
3. The following game tree is a model of the Cuban Missile Crisis of 1962. (Game theorists have produced many models of this event.)

There are two players, the Soviet Union (SU) and the United States (US). The Soviet Union’s payoff is given first at every terminal vertex. The Soviet Union initially must decide whether or not to place missiles (with nuclear warheads) in Cuba. If it places missiles in Cuba, the U.S. has three choices: attack the missile sites, blockade Cuba to put pressure on the Soviet Union to remove the missiles, or accept the missiles (let them stay in Cuba). If the U.S. attacks the missile sites, the Soviet Union can fight back or accept defeat and remove the missiles. If the U.S. blockades, the Soviet Union can accept defeat and remove the missiles, or can leave the missiles in place. If the Soviet Union leaves the missiles in place, the U.S. again faces the choice of attacking or accepting the missiles.

Note several things about the payoffs:
If the U.S. attacks and the Soviet Union fights back, payoffs are  
-10 to both, because of the danger of nuclear war.

If the Soviet Union removes the missiles before an attack, payoffs are  
-4 to the Soviet Union and 4 to the U.S.: this is a U.S. victory.

If the Soviet Union removes the missiles after an attack, payoffs are lower to both:  
-6 to the Soviet Union and 2 to the U.S. This is still a U.S. victory, but a very dangerous one for both, again  
because of the danger of nuclear war.

Use backward induction to make a recommendation as to what the  
Soviet Union should do on its first move. Make sure I can follow your  
thinking.

4. A company wants a customer to buy its product every year. The price  
of the product is 4. The value of the product to the customer is 6 if  
the product is good and 0 if it is bad. It costs the company 2 to make  
a good product and 0 to make a bad product.

We will think of this as a repeated game. In the stage game, the com-  
pany has two strategies: make a good product or make a bad product.  
The customer has two strategies: buy or don’t buy.

Suppose the company and the customer both use trigger strategies:

- The company starts by producing a good product. If the customer  
  buys in period \( k \), the company produces a good product in period  
  \( k + 1 \). If the customer does not buy in period \( k \), the company  
  produces a bad product in period \( k + 1 \) and in every subsequent  
  period.

- The customer starts by buying the company’s product. If the  
  product is good in period \( k \), the customer buys in period \( k + 1 \).  
  If the product is bad in period \( k \), the customer does not buy in  
  period \( k + 1 \) and in fact never buys again.

The discount factor is \( \delta \), \( 0 < \delta < 1 \).

(a) The payoff matrix of the stage game is

<table>
<thead>
<tr>
<th>Customer</th>
<th>buy</th>
<th>don’t buy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company</td>
<td></td>
<td></td>
</tr>
<tr>
<td>good</td>
<td>(2, 2)</td>
<td>(−2, 0)</td>
</tr>
<tr>
<td>bad</td>
<td>(4, −4)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Explain the \( (2, 2) \) entry.
(b) Find a number $\delta_0$ such that if $\delta > \delta_0$, it is a Nash equilibrium for both company and customer to use the trigger strategy.

5. When certain birds find a recently deceased small animal, they eat it. If they all eat, a crow may surprise them and drive them off. However, if at least one bird stays watchful, it can scare off any crows that may happen by. The watchful birds get to eat less than the other birds, and incur an energy cost.

For simplicity, we assume that two birds find a recently deceased animal. Each has two strategies, watch or eat. The payoffs are as follows.

<table>
<thead>
<tr>
<th>Bird 1</th>
<th>watch</th>
<th>eat</th>
</tr>
</thead>
<tbody>
<tr>
<td>watch</td>
<td>(2, 2)</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>eat</td>
<td>(3, 2)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

(a) Find the pure strategy Nash equilibria. (Note that they are not symmetric.)

(b) Find all mixed strategy Nash equilibria. (There is one and it is symmetric.)

(c) Check whether your mixed strategy Nash equilibrium is evolutionarily stable.

(d) Derive the replicator equation and reduce it to a single differential equation.

(e) Find all equilibria of your differential equation.

(f) Draw the phase portrait of your differential equation.

6. Let’s complicate the previous problem by assuming that the birds have three strategies:

- watch ($w$)
- eat ($e$)
- mixed ($m$): watch when you sense that the other bird is a greedy eater who never watches; otherwise eat.

This time we’ll assume the payoffs are as follows:

<table>
<thead>
<tr>
<th>Bird 1</th>
<th>w</th>
<th>e</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>(1, 1)</td>
<td>(0, 2)</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>e</td>
<td>(2, 0)</td>
<td>(-1, -1)</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>m</td>
<td>(2, 0)</td>
<td>(0, 2)</td>
<td>(-1, -1)</td>
</tr>
</tbody>
</table>
(a) Find the pure strategy Nash equilibria. (Note that they are not symmetric.)

(b) Find a mixed strategy Nash equilibrium in which both birds use all three strategies with positive probability. (Because of the symmetry of the problem, once you have found the three probabilities for one bird, you may assume that the other bird uses the same three probabilities.)

(c) Try to find a Nash equilibrium in which both birds use strategies $e$ and $m$ with positive probability, and strategy $w$ with 0 probability. (Again, because of the symmetry of this problem, once you have found the two probabilities for one bird, you may assume that the other bird uses the same two probabilities.) Don’t forget the final step in checking that you really have a Nash equilibrium.

(d) Denote a population type by $\sigma = pw + qe + rm$ with $p \geq 0, q \geq 0, \ r \geq 0, \ p + q + r = 1$. The replicator equation, using the variables $p$ and $q$ only, is

\[
\dot{p} = (p - h(p, q))p, \\
\dot{q} = (2 - 3q - h(p, q))q,
\]

with $h(p, q) = -1 + 4p + 4q - 2p^2 - 4pq - 4q^2$. Find three equilibria of this equation for which $p = 0$.

(e) Compute the eigenvalues of the linearization at the equilibrium $(0, \frac{3}{4})$ and describe its type (attractor, repeller, saddle).

(f) There are actually seven equilibria:

- the corners of the triangle, each of which is a repeller;
- three more points on the boundary of the triangle, each of which is a saddle;
- the interior equilibrium, which is an attractor

The phase portrait is shown below. What does it tell you?