1. Let
\[ f(x, y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \]

Using the definition of derivative, prove: \( Df(0, 0) = [0 \ 0] \). Hint: \( x^2 \leq x^2 + y^2 \).

2. Suppose \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) is
\[ f(x_1, x_2, x_3) = (x_1^2 x_2 x_3, x_2^2 + x_3^2), \]
and \( g : \mathbb{R} \to \mathbb{R}^3 \) is
\[ g(t) = (2t + 1, e^t, 4). \]

(a) Calculate \( Df(x_1, x_2, x_3) \) and \( Dg(t) \).
(b) Using the chain rule that we learned this semester, which involves multiplication of matrices, calculate \( D(f \circ g)(0) \).

3. Consider the system of equations
\[ (x^2 + u^2)(y^2 + v^2) = 1, \]
\[ x \cos u + y \sin v = 1. \]

(a) Show that the Implicit Function Theorem implies we can solve for \((x, y)\) in terms of \((u, v)\) near \((x, y, u, v) = (1, 1, 0, 0)\).
(b) Compute the matrix of partial derivatives of \((x, y)\) with respect to \((u, v)\) at that point.
Do two of the following three problems.

(4) Prove: If \( f: \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( x_0 \), then \( f \) is continuous at \( x_0 \).

(5) Suppose \( f: \mathbb{R}^n \to \mathbb{R} \) is a \( C^2 \) function, \( Df(0) = 0 \), and the bilinear form associated with \( D^2f(0) \) is negative definite. Prove that \( f \) has a local maximum at \( x = 0 \). (In your proof you may use Taylor’s formula and the lemma that says: if \( B \) is a negative definite bilinear form, then there is a constant \( m > 0 \) such that \( B(x, x) \leq -m \|x\|^2 \) for all \( x \in \mathbb{R}^n \)).

(6) Let \( A \subset \mathbb{R}^n \) be convex. (This means that if \( x \) and \( y \) are in \( A \), then the entire line segment that joins them is in \( A \).) Let \( f: A \to \mathbb{R}^m \) be \( C^1 \). Suppose there is a number \( M > 0 \) such that for all \( x \in A \) and all \( z \in \mathbb{R}^n \), \( \|Df(x)z\| \leq M\|z\| \). Prove: If \( x \in A \) and \( y \in A \) then \( \|f(x) - f(y)\| \leq M\|y - x\| \). Suggestions: (1) Let \( h(t) = f(x+t(y-x)) \). (2) You may use the following: If \( g: [a, b] \to \mathbb{R}^m \) is continuous, then \( \| \int_a^b g(t) dt \| \leq \int_a^b \| g(t) \| dt \).