1. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Show \( f \) is continuous at \((0, 0)\). Suggestion: \(|x| \leq \sqrt{x^2 + y^2}\) and \(|y| \leq \sqrt{x^2 + y^2}\).

(b) Find all directional derivatives of \( f \) at \((0, 0)\) that exist.

(c) Is \( f \) differentiable at \((0, 0)\)? Justify your answer.

2. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(x, y) = \begin{cases} 
x^2 y & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Show that all directional derivatives of \( f \) at \((0, 0)\) exist.

(b) Show that \( f \) not continuous at \((0, 0)\). Suggestion: Look at the values of \( f \) on the curve \( \phi(t) = (t, t^2) \), which goes through the origin.

3. Consider the function \( f : \mathbb{R}^2 \to \mathbb{R} \) given by

\[
f(x, y) = \begin{cases} 
\frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

(a) Show that \( f \) is differentiable at \((0, 0)\) and \( Df(0, 0) = 0 \).

(b) Calculate \( \frac{\partial f}{\partial x}(0, y) \). In other words, at each point on the \( y \)-axis, find the partial derivative of \( f \) with respect to \( x \). (Unless \( y = 0 \), you can just do this using the quotient rule.)
(c) Calculate $\frac{\partial f}{\partial y}(x, 0)$. In other words, at each point on the $x$-axis, find the partial derivative of $f$ with respect to $y$. (Unless $x = 0$, you can just do this using the quotient rule.)

(d) Using your answers to parts (b) and (c) show that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

(e) For this function, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist at every point $(x, y)$. Do you think that they are continuous at $(0, 0)$? Hint: Equality of mixed partials.

4. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called homogeneous of degree $n$ if for all $(x, y)$ in $\mathbb{R}^2$ and for all $t > 0$,

$$f(tx, ty) = t^n f(x, y).$$

(a) Show that the functions $f(x, y) = \frac{xy}{x+y}$, $f(x, y) = \frac{x}{x^2+y^2}$, and $f(x, y) = x^{1/3} + x^{-2/3}y$ are all homogeneous. What are the degrees?

(b) Prove Euler’s Theorem: If $f : \mathbb{R}^2 \to \mathbb{R}$ is homogeneous of degree $n$, then at any point $(x, y)$ where $f$ is differentiable,

$$xf_x(x, y) + yf_y(x, y) = nf(x, y).$$

Suggestion. Fix a point $(x, y)$ where $f$ is differentiable, and define $\phi : \mathbb{R} \to \mathbb{R}^2$ by $\phi(t) = (tx, ty)$. (Remember, $(x, y)$ is fixed.) Consider the composition $f \circ \phi(t) = f(tx, ty)$. Calculate the derivative of $f \circ \phi$ at $t = 1$ two different ways, one with the chain rule, one without the chain rule.