Material to Review for the MA 426-591M Final

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In the following, when I say you should know a definition or the statement of something, that doesn't mean you should know it word-for-word; it just means that this is something that you should be able to use. When I say you should know a proof, again I don't mean word-for-word, but that you should be able to prove this thing if asked. The problems listed are ones you should be able to do.

Whenever the book mentions metric spaces in definitions, theorems, etc., you can substitute $\mathbb{R}^n$.

1. Topology of $\mathbb{R}^n$.

   - 1.6 and 1.7. Euclidean space, norms, inner products, metrics. Consider these sections as background.
   - 2.1. Open sets. Definition 2.1.1; proof of Proposition 2.1.2; proofs that $\{x \in \mathbb{R}^n : \| x \| > 1\}$, $\{x \in \mathbb{R}^2 : x_1 > 0\}$, and $\{x \in \mathbb{R}^2 : x_1 < 1\}$ are open; proof of Proposition 2.1.3; problems 2.1.1 and 2.1.3.
   - 2.2. Interior of a set. Omit.
   - 2.3. Closed sets. Definition 2.3.1, statement of Proposition 2.3.2. Proofs from lecture that various sets are closed.
   - 2.4. Accumulation points. Definition 2.4.1, statement of Theorem 2.4.2; second problem assigned Aug. 26 (see http://courses.ncsu.edu/ma426/lec/003/assign.html).
   - 2.5. Closure. Definition 2.5.1, statement of Proposition 2.5.2, problem 2.5.2.
   - 2.6. Boundary. Definition 2.6.1, proof of Proposition 2.6.2, Example 2.6.3.
   - 2.7. Sequences. Definition 2.7.1, statement of Propositions 2.7.2 and 2.7.3, proof of Proposition 2.7.4, statement of Proposition 2.7.6, example 2.7.7, problem 2.7.3 as we did it.
   - 2.8. Completeness. You should know what a Cauchy sequence is and the statement of Theorem 2.8.5.

- 2.9. Infinite series. Omit.

2. Compactness, path-connectedness, continuity.

   - 3.1–3.3. Compactness. The following material is from lectures (we didn't follow the text).
     (a) Definition of a bounded set.
     (b) Statement of Nested Sets Property as done in lecture.
     (c) Statement of Bolzano-Weierstrass Theorem as done in lecture.
     (d) Open cover definition of compactness, problems 1–3, 5, 6 assigned Sept. 13 (see http://courses.ncsu.edu/ma426/lec/003/assign.html).
   - 4.1. Continuity. Definition 4.1.1, the rephrasing of Definition 4.1.2, and Definition 4.1.3 (especially the second sentence); statement from lecture of "Easy Theorem 4.1.4" and proofs that (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i), and (i) $\Rightarrow$ (iii) for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$; proof that $f(x_1, x_2) = x_1$ is continuous, and use of this result to do problem 3 on p. 108; problem 4.1.3.
   - 4.2. Images of compact sets under continuous maps. Proof of Theorem 4.2.1 (both the convergent subsequence proof and the open cover proof).

     item 4.3 Operations on continuous mappings. Proof of Theorem 4.3.1 for $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$, statements of Proposition 4.3.2 and Corollary 4.3.3.
   - 4.4. Boundedness of continuous functions on compact sets. Statement of Theorem 4.4.1.
   - 3.4, 4.2, 4.5. Path-connected sets. Definition, proof that $D(0, \epsilon)$ is path connected, proof of Theorem 4.2.1 for path-connected (not connected) sets. Problem 4.2.1, 4.5.2.
   - 4.6. Uniform continuity. Definition 4.6.1, statement of Theorem 4.6.2.
   - 5.1. Pointwise and uniform convergence. Definitions 5.1.1 and 5.1.2; the lemma proved in lecture for showing that a sequence f_k converges uniformly; proof of Proposition 5.1.4; problems 5.1.1, 5.1.2, 5.1.3.

3. The derivative.

   - 6.1. Definition of the derivative. Definition 6.1.1. Problems 6.1.2 and 6.1.4; p. 384 problem 2; proof that if $f(x) = Ax + b$ then $DF(x) = A$.
   - 6.2. Matrix representation. Definition 6.2.1. Statement of Theorem 6.2.2. Problems 6.2.1, 6.2.2.
   - 6.3. Continuity of differentiable functions. Proofs of the following: (1) if $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there is a constant M such that $\| L(x) \| \leq M \| x \|$; (2) if $f : A \to \mathbb{R}^m$ is differentiable at x_0, then f is continuous at x_0.
   - 6.4. Conditions for differentiability. The example that begins the section (figure 6.4.1). Statement of Theorem 6.4.1. Definition 6.4.2. Example 6.4.3. Problem 6.4.2 as we did it.

- 6.5. Differentiation rules. Proof of the "multiplication by a constant rule" and the "sum rule." Statement of the chain rule (Theorem 6.5.1). Relation of the chain rule to the directional derivative formula (Example 6.5.4 as done in lecture). Problems 6.5.2, 6.5.3.
- 6.7. Mean value theorem. Proof of Theorem 6.7.1 (first part). Problem 6.7.5.
- 6.8. Second derivative and Taylor's Theorem. The idea of the derivative of Df. Statements of Theorems 6.8.2 and 6.8.3, and of the version of Theorem 6.8.5 that we did in lecture. Problem 6.8.5.
- 6.9. Maxima and minima. Statement of Theorem 6.9.2. Proofs of Theorem 6.9.4, and of the lemma we used to prove it. Problems 6.9.3, 6.9.6.

4. Inverse and implicit function theorems.

   - 7.1. Inverse Function Theorem. Statement of Theorem 7.1.1. Problems 7.1.1, 7.1.2, 7.1.5, and p. 442 problem 25.
   - 7.2. Implicit Function Theorem. Statement of Theorem 7.2.1. Pp. 439–440 problems 4, 6, 8.
   - 7.7. Lagrange multipliers. Omit.

5. Integration.

   - 8.1. Basic definitions, statement of Riemann's Criterion (8.1.3).
   - 8.2. Volume. Statement of equivalent condition for volume 0 given in lecture. Proof that the graph of a continuous function $f : I \to \mathbb{R}$ has volume 0. Problems 2 and 6 assigned Nov. 19 (see http://courses.ncsu.edu/ma426/lec/003/assign.html).
   - 8.3. Statement of our simplified version of Lebesgue's Theorem.
   - 8.4. Statement of properties (i) to (v) of the integral. Problems 3, 4, 5 assigned Nov. 19.
   - 9.3. Change of variables. Omit.