MA 425-002 Final Exam

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Do eight problems. The answers to problems 1, 3, 4, 5 and 8 should include the expression, “Let \( \epsilon > 0. \)

1. Let \( x_n = \frac{2n^2}{1+n^2}. \) Prove that \( x_n \to 2. \)

2. Let \( (x_n) \) be a sequence such that \( x_n \to x. \) Suppose \( x < 0. \) Prove that there is a number \( N \) such that \( x_n < 0 \) for all \( n > N. \)

3. Prove: If \( (x_n) \) is a bounded decreasing sequence and \( u = \inf \{x_n : n \in \mathbb{N} \}, \) then \( x_n \to u. \)

4. Let \( f : (0, \infty) \to \mathbb{R} \) and \( g : (0, \infty) \to \mathbb{R} \) be functions. Assume:
   
   (a) \( f(x) > 0 \) for all \( x. \)
   (b) \( \lim_{x \to 0} f(x) = \infty. \)
   (c) \( g \) is a bounded function.

   Prove that \( \lim_{x \to 0} \frac{g(x)}{f(x)} = 0. \)

5. Show that the function \( f(x) = \frac{1+x}{x^2} \) is uniformly continuous on the interval \( 1 \leq x < \infty. \)

6. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that for every \( x, -x^2 \leq f(x) \leq x^2. \) Prove that \( f \) is differentiable at \( 0, \) and \( f'(0) = 0. \) (Notice that \( f(0) \) has to be 0. Be careful with this problem. If you divide by \( x \) when \( x \) is negative, inequalities reverse.)
7. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function. Assume that \( f' \) is a strictly increasing function on \([a, b]\). (This means: If \( x_1 \in [a, b], x_2 \in [a, b], \) and \( x_1 < x_2 \), then \( f'(x_1) < f'(x_2) \).) Prove: \( f(b) - f(a) - f'(a)(b - a) > 0 \). Hint: What does the Mean Value Theorem tell you about \( f(b) - f(a) \)?

8. Let

\[
\begin{align*}
f_n(x) &= \frac{1 + nx^2}{nx}, \quad 0 < x < \infty, \\
f(x) &= x, \quad 0 < x < \infty.
\end{align*}
\]

Show that if \( a > 0 \), then \( f_n \to f \) uniformly on the interval \( a \leq x < \infty \).

9. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function, and let \( c \in (a, b) \). Assume:

- \( f(x) \geq 0 \) for all \( x \in [a, b] \).
- \( f(c) > 0 \).

Show that \( \int_a^b f > 0 \).

10. Let \( (a_n) \) and \( (b_n) \) be positive sequences. Suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) both converge. Show that \( \sum_{n=1}^{\infty} a_n b_n \) converges.