

# MA 425-003 Final Exam

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Do eight problems. The answers to all problems except 7 and 8 must use an  $\epsilon$ .

1. Let  $x_n = 1 + \frac{2n}{1+n^2}$ . Prove that  $x_n \rightarrow 1$ .
2. Let  $(x_n)$  be a sequence such that (1)  $x_n > 1$  for every  $n$  and (2)  $x_n \rightarrow x$ . Prove that  $x \geq 1$ .
3. Prove: If  $(x_n)$  is a bounded decreasing sequence and  $u = \inf\{x_n : n \in \mathbb{N}\}$ , then  $x_n \rightarrow u$ .
4. Let  $A \subset \mathbb{R}$ , let  $c$  be a cluster point of  $A$ , and let  $f$  and  $g$  be functions from  $A$  to  $\mathbb{R}$ . Suppose  $\lim_{x \rightarrow c} f(x) = a$  and  $\lim_{x \rightarrow c} g(x) = b$ . Prove that  $\lim_{x \rightarrow c} f(x) - g(x) = a - b$ .
5. Let  $f : (c, \infty) \rightarrow \mathbb{R}$  and  $g : (c, \infty) \rightarrow \mathbb{R}$  be functions. Assume (1)  $f(x) > 0$  for all  $x$ , (2)  $\lim_{x \rightarrow c} f(x) = \infty$ , and (3)  $g$  is a bounded function. Prove that  $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$ .
6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be uniformly continuous functions. Prove that  $g \circ f$  is uniformly continuous.
7. Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded function and let  $f(x) = x^2g(x)$ . Prove that  $f'(0) = 0$ . (Don't use the product rule for differentiation. We don't know that  $g$  is differentiable at 0.)
8. Let  $f$  be continuously differentiable on  $[a, b]$ . (This means that  $f'(x)$  exists for every  $x \in [a, b]$ , and  $f'$  is itself a continuous function.) Assume that  $f'(x) < 0$  for every  $x \in [a, b]$ . Let  $k = \sup\{f'(x) : x \in [a, b]\}$ .

- (a) Is  $k$  positive, negative, or 0? Explain. (Hint: What do we know about continuous functions on closed intervals?)
- (b) Use the Mean Value Theorem to prove that if  $x_1$  and  $x_2$  are any two points of  $[a, b]$  with  $x_1 < x_2$ , then the slope of the line segment from  $(x_1, f(x_1))$  to  $(x_2, f(x_2))$  is less than or equal to  $k$ .

9. Define  $f : [0, 2] \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 2, \\ 3 & \text{if } x = 2. \end{cases}$$

Prove that  $f$  is Riemann integrable on  $[0, 2]$  and  $\int_0^2 f = 0$ .

10. Show that if  $a > 0$ , then as  $n \rightarrow \infty$ ,  $\frac{nx}{1+nx} \rightarrow 1$  uniformly on  $[a, \infty)$ .