

STEP FUNCTIONS, DELTA FUNCTIONS, AND THE VARIATION OF PARAMETERS FORMULA

STEPHEN SCHECTER

1. THE UNIT STEP FUNCTION AND PIECEWISE CONTINUOUS FUNCTIONS

The *Heaviside unit step function* $u(t)$ is given by

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t > 0. \end{cases}$$

The function $u(t)$ is not defined at $t = 0$. Often we will not worry about the value of a function at a point where it is discontinuous, since often it doesn't matter.

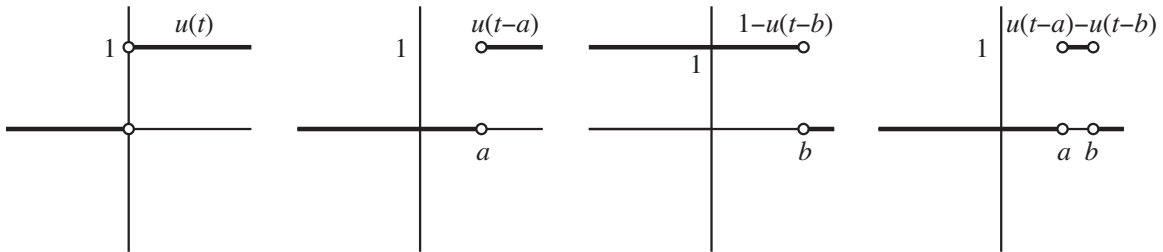


FIGURE 1.1. Heaviside unit step function.

The function $u(t)$ turns on at $t = 0$. The function $u(t - a)$ is just $u(t)$ shifted, or dragged, so that it turns on at $t = a$.

The function $1 - u(t - b)$ turns off at $t = b$.

The function $u(t - a) - u(t - b)$, with $a < b$, turns on at $t = a$ and turns off at $t = b$.

We can use the unit step function to crop and shift functions. Multiplying $f(t)$ by $u(t - a)$ crops $f(t)$ so that it turns on at $t = a$:

$$u(t - a)f(t) = \begin{cases} 0 & \text{if } t < a, \\ f(t) & \text{if } t > a. \end{cases}$$

Multiplying $f(t)$ by $u(t - a) - u(t - b)$, with $a < b$, crops $f(t)$ so that it turns on at $t = a$ and turns off at $t = b$:

$$(u(t - a) - u(t - b))f(t) = \begin{cases} 0 & \text{if } t < a, \\ f(t) & \text{if } a < t < b, \\ 0 & \text{if } t > b. \end{cases}$$

We can also crop $f(t)$ at $t = 0$ and drag it to $t = a$:

$$u(t-a)f(t-a) = \begin{cases} 0 & \text{if } t < a, \\ f(t-a) & \text{if } t > a. \end{cases}$$

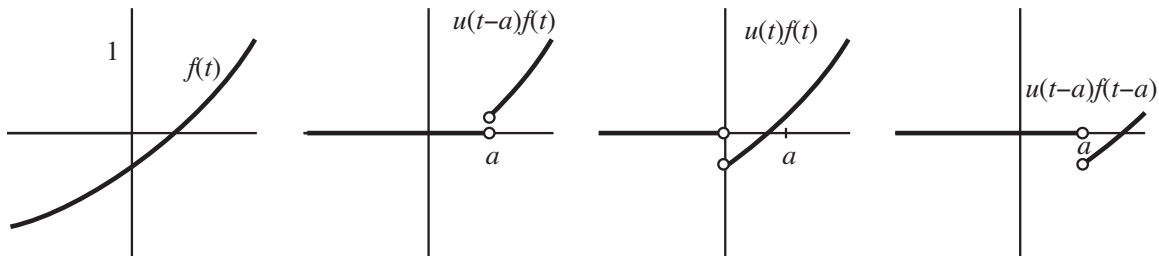


FIGURE 1.2. Cropping and dragging.

A function $f(t)$ is said to have a *jump discontinuity* at $t = a$ if

- (1) there are numbers $t_0 < a < t_1$ such that f is defined and continuous on $[t_0, a)$ and on $(a, t_1]$;
- (2) $\lim_{t \rightarrow a^-} f(t)$ and $\lim_{t \rightarrow a^+} f(t)$ both exist as finite numbers; and
- (3) $\lim_{t \rightarrow a^-} f(t) \neq \lim_{t \rightarrow a^+} f(t)$.

A function $f(t)$ is called *piecewise continuous* on the real line if

- (1) f is defined and continuous except at certain points where it has a jump discontinuity, and
- (2) the number of jump discontinuities in any closed interval is finite.

Usually we leave piecewise continuous functions undefined at the points where they have jump discontinuities, because usually their values there don't matter.

We will use the notation

$$f(a-) = \lim_{t \rightarrow a^-} f(t), \quad f(a+) = \lim_{t \rightarrow a^+} f(t).$$

These numbers are the same at a point a where f is continuous, but they are different at a point a where f has a jump discontinuity.

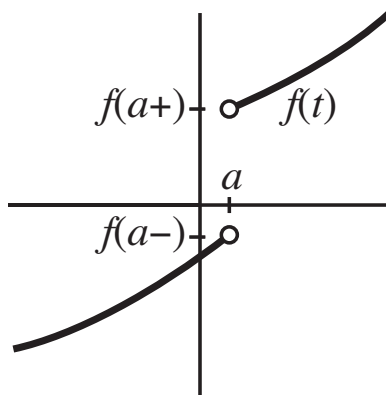


FIGURE 1.3. A piecewise continuous function.

The functions pictured so far are piecewise continuous on the real line.

2. THE DIRAC DELTA FUNCTION

Piecewise continuous functions have continuous antiderivatives. For example, an antiderivative of $u(t)$ is the “unit ramp function”

$$\int u(t) dt = \begin{cases} 0 & \text{if } t < 0, \\ t & \text{if } t \geq 0. \end{cases}$$

A function with a jump discontinuity cannot possibly have a derivative at that point. Nevertheless, let’s ask the question: if the unit step function $u(t)$ had a derivative $u'(t)$, what would its properties be?

Actually, wherever $t \neq 0$, $u(t)$ has a derivative, and it’s 0. So the first property is:

$$(1) u'(t) = 0 \text{ for } t \neq 0.$$

The second property comes from the Fundamental Theorem of Calculus: if $t_0 < 0 < t_1$, we should have

$$\int_{t_0}^{t_1} u'(t) dt = u(t)]_{t_0}^{t_1} = u(t_1) - u(t_0) = 1 - 0 = 1.$$

Thus our second property is:

$$(2) \text{ If } t_0 < 0 < t_1, \text{ then } \int_{t_0}^{t_1} u'(t) dt = 1.$$

No ordinary function has properties (1) and (2)! Nevertheless, we will go ahead and define a “generalized function” that does have these properties. It’s called the *Dirac delta function* $\delta(t)$, and its properties are:

$$(1) \delta(t) = 0 \text{ for } t \neq 0.$$

$$(2) \text{ If } t_0 < 0 < t_1, \text{ then } \int_{t_0}^{t_1} \delta(t) dt = 1.$$

We say $\delta(t)$ is *concentrated* at $t = 0$.

There are certain operations that one can legitimately do with $\delta(t)$. One can shift it: $\delta(t - a)$ is concentrated at $t = a$. One can also multiply $\delta(t - a)$ by a function $f(t)$ that is continuous on an open interval that contains $t = a$. The resulting product has the following properties:

$$(1) \delta(t - a)f(t) = 0 \text{ for } t \neq a.$$

$$(2) \text{ If } t_0 < a < t_1, \text{ then } \int_{t_0}^{t_1} \delta(t - a)f(t) dt = f(a).$$

Notice that the value of the integral only depends on the value of f at $t = a$. A consequence of property (2) is:

$$\text{If } t_0 < a < t_1, \text{ then } \int_{t_0}^{t_1} k\delta(t - a) dt = k.$$

There are other operations that I do not recommend trying with $\delta(t)$. For example, you should not try to square it, and you should not multiply it by a function that has a jump discontinuity at $t = 0$.

3. FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

Piecewise continuous functions and delta functions can be used as forcing terms in linear differential equations. We’ll just consider linear differential equations with constant coefficients.

We’ll often consider *rest initial conditions*:

$$y(t) = 0 \text{ for } t < 0.$$

Let's first look at first-order linear differential equations

$$(3.1) \quad my' + ky = f(t).$$

Rules about solutions of (3.1):

- (1) If $f(t)$ is piecewise continuous, any solution $y(t)$ is continuous.
- (2) If $f(t)$ is a delta function, any solution $y(t)$ has a jump discontinuity where the delta function is concentrated.

An important consequence of rule (2) is the following:

Theorem 3.1. *Any solution $y(t)$ of*

$$(3.2) \quad my' + ky = \delta(t - a)$$

has $y(a+) - y(a-) = \frac{1}{m}$.

Here is why this formula for the jump is correct. Let $t_0 < a < t_1$, let $y(t)$ be a solution of (3.2), and integrate both sides of (3.2) from $t = t_0$ to $t = t_1$. We get

$$m(y(t_1) - y(t_0)) + k \int_{t_0}^{t_1} y(t) dt = 1.$$

Now let t_0 and t_1 approach a . By rule (2), $y(t)$ is an ordinary piecewise continuous function. When we integrate it over shorter and shorter intervals, the integral approaches 0. On the other hand, $y(t_1) - y(t_0)$ approaches $y(a+) - y(a-)$. Passing to the limit, we have

$$m(y(a+) - y(a-)) = 1.$$

Therefore

$$y(a+) - y(a-) = \frac{1}{m}.$$

Example. A new territory is discovered. One million people immediately move there. Thereafter the population grows naturally with a growth rate of 4% per year. What is the population after t years?

Solution 1. Let p denote the population in millions. Due to the immigration, the population at time 0 is 1. Thus we have

$$\frac{dp}{dt} = 0.04p, \quad p(0) = 1.$$

The solution of this initial value problem is $p = e^{0.04t}$. We should only use this formula for $t \geq 0$.

Solution 2. Let p denote the population in millions. Then

$$\frac{dp}{dt} = 0.04p + \delta(t), \quad p(t) = 0 \text{ for } t < 0.$$

Since the forcing function is a delta function concentrated at $t = 0$, by rule (2) $p(t)$ has a jump discontinuity at $t = 0$. By Theorem 3.1 the jump is $p(0+) - p(0-) = \frac{1}{1} = 1$. Since $p(0-) = 0$, we have $p(0+) = 1$. Therefore, for $t > 0$ we solve the initial value problem

$$\frac{dp}{dt} = 0.04p, \quad p(0) = 1$$

exactly as before. The solution to our problem is

$$p(t) = \begin{cases} 0 & \text{if } t < 0, \\ e^{-0.04t} & \text{if } t > 0. \end{cases}$$

4. SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

Now let's consider second-order linear differential equations

$$(4.1) \quad my'' + by' + ky = f(t).$$

Rules about solutions of (4.1):

- (1) If $f(t)$ is piecewise continuous and $y(t)$ is a solution, then $y(t)$ is differentiable and $y'(t)$ is continuous.
- (2) If $f(t)$ is a delta function and $y(t)$ is a solution, then $y(t)$ is continuous, but $y'(t)$ has a jump discontinuity where the delta function is concentrated.

An important consequence of rule (2) is the following:

Theorem 4.1. Any solution $y(t)$ of

$$(4.2) \quad my'' + by' + ky = \delta(t - a)$$

has $y'(a+) - y'(a-) = \frac{1}{m}$.

To derive this formula, let $t_0 < a < t_1$ and integrate both sides of (4.2) from $t = t_0$ to $t = t_1$. We get

$$m(y'(t_1) - y'(t_0)) + b(y(t_1) - y(t_0)) + k \int_{t_0}^{t_1} y(t) dt = 1.$$

Now let t_0 and t_1 approach a . By rule (2), $y(t)$ is a continuous function, so $y(t_1) - y(t_0)$ and the integral approach 0. On the other hand, $y'(t_1) - y'(t_0)$ approaches $y'(a+) - y'(a-)$. Passing to the limit, we have

$$m(y'(a+) - y'(a-)) = 1.$$

Therefore

$$y'(a+) - y'(a-) = \frac{1}{m}.$$

When a force $f(t)$ is applied to a mass from time $t = t_0$ to time $t = t_1$, a change in momentum results:

$$mv(t_1) - mv(t_0) = \int_{t_0}^{t_1} m \frac{dv}{dt} dt = \int_{t_0}^{t_1} f(t) dt.$$

The integral $\int_{t_0}^{t_1} f(t) dt$ is called the *impulse* due to the force. It is the change in momentum that the force produces.

Suppose a mass m is at rest and we strike it at time $t = 0$ with a hammer. The mass takes off with velocity v_1 . Clearly the hammer imparted a force over a very short time that caused the change in momentum. How do we represent this force mathematically?

Let's make a mathematical idealization: $v(t) = 0$ for $t < 0$ and $v(t) = v_1$ for $t > 0$. Let $f(t)$ be the force imparted by the hammer. Then for any $t_0 < 0 < t_1$, we have

$$(4.3) \quad \int_{t_0}^{t_1} f(t) dt = mv(t_1) - mv(t_0) = mv_1 - 0 = mv_1.$$

Therefore $f(t) = mv_1\delta(t)$.

In mechanics problems in which a force is imparted instantaneously at time $t = a$, the force is represented mathematically by a constant times $\delta(t - a)$. The constant represents the impulse due to the force. Such an instantaneously applied force is often itself called an impulse.

Example. A spring-mass system is at rest. We strike the mass with a hammer, hard enough to cause an instantaneous positive change in momentum of 1 unit. What is the response?

Solution 1. After $t = 0$, the differential equation is

$$my'' + by' + ky = 0.$$

The system starts at rest so $y(0) = 0$. Because of the hammer blow, we instantly get a momentum of 1, so we might as well take $my'(0) = 1$, i.e., $y'(0) = \frac{1}{m}$. Now we have an initial value problem to solve. Call the solution $h(t)$. Of course, we only use $h(t)$ for $t \geq 0$.

Solution 2. The differential equation and initial conditions are

$$my'' + by' + ky = \delta(t), \quad y(t) = 0 \text{ for } t < 0.$$

Recall our rule for solving second-order linear differential equations with a delta function forcing term: the solution is continuous, and its first derivative has a jump discontinuity where the delta function is concentrated. Since the solution is continuous, we have $y(0) = 0$. By Theorem 4.1, the jump in the derivative is $y'(0+) - y'(0-) = \frac{1}{m}$. Since $y'(0-) = 0$, we have $y'(0+) = \frac{1}{m}$. Therefore, for $t > 0$, we solve the initial value problem

$$my'' + by' + ky = 0, \quad y(0) = 0, \quad y'(0) = \frac{1}{m}$$

as before. The solution to our problem is

$$y(t) = \begin{cases} 0 & \text{if } t < 0, \\ h(t) & \text{if } t \geq 0. \end{cases}$$

5. VARIATION OF PARAMETERS FORMULA FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

Definition. The solution to the problem

$$y' + ay = \delta(t), \quad y(t) = 0 \text{ for } t < 0.$$

is called the *impulse response function* for the differential operator $y' + ay$. It is given by

$$y(t) = \begin{cases} 0 & \text{if } t < 0, \\ e^{-at} & \text{if } t \geq 0. \end{cases}$$

Theorem 5.1. *The solution of*

$$(5.1) \quad y' + ay = f(t), \quad y(0) = y_0$$

is

$$(5.2) \quad y(t) = y_0 e^{-at} + \int_0^t e^{-a(t-s)} f(s) ds.$$

Formula (5.2) is called the *variation of parameters formula*. The reason for this name is not important.

We will give two derivations of (5.2).

Derivation 1. Solve (5.1) using the method we learned in Section 2.3. Multiply (5.1) by e^{at} , rewrite the left-hand side as a derivative, and integrate:

$$\begin{aligned} e^{at}y' + ae^{at}y &= e^{at}f(t), \\ \frac{d}{dt}(e^{at}y) &= e^{at}f(t), \\ e^{at}y &= \int_0^t e^{as}f(s) ds + c. \end{aligned}$$

To integrate the right-hand side, we had to choose an antiderivative of $e^{at}f(t)$; we chose $\int_0^t e^{as}f(s) ds$. Multiply by e^{-at} :

$$y = ce^{-at} + e^{-at} \int_0^t e^{as}f(s) ds = ce^{-at} + \int_0^t e^{-a(t-s)}f(s) ds.$$

Set $t = 0$ and $y = y_0$ to determine c : $c = y_0$.

Derivation 2. This derivation uses the impulse response function. It should give you a good intuitive understanding of why the formula is true.

To estimate $y(t)$ at some $t > 0$, divide the interval $[0, t]$ into n equal parts of length h : $0 = t_0 < t_1 < \dots < t_n = t$. The impulse given to the system between $t = t_{i-1}$ and $t = t_i$ is $\int_{t_{i-1}}^{t_i} f(t) dt$, which is approximately $f(t_i)h$. Instead of applying the continuous force $f(t)$, let's apply the force in discrete impulses at the times t_i . Then our force will be

$$f_h(t) = \sum_{i=1}^n \delta(t - t_i)f(t_i)h.$$

If h is small, the force $f_h(t)$ should produce approximately the same result as $f(t)$.

The response to the impulse $\delta(t - t_i)f(t_i)h$ is of course $e^{-a(t-t_i)}f(t_i)h$, $t \geq t_i$. (The stimulus response function is shifted to $t = t_i$ and multiplied by the appropriate constant.) To get the response to $f_h(t)$ at the particular time t mentioned earlier, the responses to the little blows at all the times t_i must be added. Therefore the response is

$$y_h(t) = \sum_{i=1}^n e^{-a(t-t_i)}f(t_i)h.$$

This is a Riemann sum for the integral $\int_0^t e^{-a(t-s)}f(s) ds$. Therefore, as $h \rightarrow 0$, $y_h(t)$ approaches $\int_0^t e^{-a(t-s)}f(s) ds$. This is therefore the response $y(t)$ of the system to the continuous forcing function $f(t)$.

Actually, this is the response when the initial condition is $y(0) = 0$. It is therefore a particular solution of (5.1). To get the general solution, add the general solution of the homogeneous problem, ce^{-at} , and determine c from the initial condition.

6. VARIATION OF PARAMETERS FORMULA FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

Definition. The solution to the problem

$$my'' + by' + ky = \delta(t), \quad y(t) = 0 \text{ for } t < 0$$

is called the *impulse response function* for the differential operator $my'' + by' + ky$. It is given by

$$y(t) = \begin{cases} 0 & \text{if } t < 0, \\ h(t) & \text{if } t \geq 0, \end{cases}$$

where $h(t)$ is the solution of the initial value problem

$$my'' + by' + ky = 0, \quad y(0) = 0, \quad y'(0) = \frac{1}{m}.$$

Sometimes we will call $h(t)$, defined for $t \geq 0$, the impulse response function. I hope this won't be confusing.

Theorem 6.1. *The solution of*

$$(6.1) \quad my'' + by' + ky = f(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

is

$$(6.2) \quad y(t) = y_c(t) + \int_0^t h(t-s)f(s) ds.$$

In this formula, $y_c(t)$ is the solution of the homogeneous initial value problem

$$my'' + by' + ky = 0, \quad y(0) = y_0, \quad y'(0) = y_1,$$

and $h(t)$ is the impulse response function.

Formula (6.2) is another version of the variation of parameters formula.

We will only give one derivation of (6.2). It almost word-for-word the same as our derivation of formula (5.2) using the impulse response function.

To estimate $y(t)$ at some $t > 0$, divide the interval $[0, t]$ into n equal parts of length h : $0 = t_0 < t_1 < \dots < t_n = t$. The impulse given to the system between $t = t_{i-1}$ and $t = t_i$ is $\int_{t_{i-1}}^{t_i} f(t) dt$, which is approximately $f(t_i)h$. Instead of applying the continuous force $f(t)$, let's apply the force in discrete impulses at the times t_i . Then our force will be

$$f_h(t) = \sum_{i=1}^n \delta(t - t_i) f(t_i) h.$$

If h is small, the force $f_h(t)$ should produce approximately the same result as $f(t)$.

The response to the impulse $\delta(t - t_i) f(t_i) h$ is of course $h(t - t_i) f(t_i) h$, $t \geq t_i$. (The stimulus response function is shifted to $t = t_i$ and multiplied by the appropriate constant.) To get the response to $f_h(t)$ at the particular time t mentioned earlier, the responses to the little blows at all the times t_i must be added. Therefore the response is

$$y_h(t) = \sum_{i=1}^n h(t - t_i) f(t_i) h.$$

This is a Riemann sum for the integral $\int_0^t h(t-s)f(s) ds$. Therefore, as $h \rightarrow 0$, $y_h(t)$ approaches $\int_0^t h(t-s)f(s) ds$. This is therefore the response $y(t)$ of the system to the continuous forcing function $f(t)$.

Actually, this is the response when the initial condition is $y(0) = 0$. It is therefore a particular solution of (5.1). To get the general solution, add the general solution of the homogeneous problem, and determine c_1 and c_2 from the initial condition.

Example. Solve

$$\frac{d^2y}{dt^2} + y = \tan t, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution. According to Theorem 6.1, the solution is

$$(6.3) \quad y(t) = \int_0^t h(t-s) \tan s \, ds,$$

where $h(t)$ is the impulse response function.

The impulse response function $h(t)$ is the solution of

$$\frac{d^2y}{dt^2} + y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

We easily find that $h(t) = \sin t$.

From (6.3),

$$\begin{aligned} y(t) &= \int_0^t \sin(t-s) \tan s \, ds \\ &= \int_0^t (\sin t \cos s - \cos t \sin s) \tan s \, ds \\ &= \sin t \int_0^t \cos s \tan s \, ds - \cos t \int_0^t \sin s \tan s \, ds \\ &= \sin t \int_0^t \sin s \, ds - \cos t \int_0^t \frac{\sin^2 s}{\cos s} \, ds. \end{aligned}$$

Now

$$\int_0^t \sin s \, ds = -\cos s \Big|_0^t = -\cos t + 1.$$

With the help of p. 196 in the text, we find that

$$\int_0^t \frac{\sin^2 s}{\cos s} \, ds = \ln |\sec s + \tan s| - \sin s \Big|_0^t = \ln |\sec t + \tan t| - \sin t.$$

Therefore

$$(6.4) \quad \begin{aligned} y(t) &= \sin t (-\cos t + 1) - \cos t (\ln |\sec t + \tan t| - \sin t) \\ &= \sin t - \cos t \ln |\sec t + \tan t|. \end{aligned}$$

A slightly more general version of Theorem 6.1 is sometimes useful.

Theorem 6.2. *The solution of*

$$(6.5) \quad my'' + by' + ky = f(t), \quad y(a) = y_0, \quad y'(a) = y_1$$

is

$$(6.6) \quad y(t) = y_c(t) + \int_a^t h(t-s)f(s) \, ds.$$

In this formula, $y_c(t)$ is the solution of the homogeneous initial value problem

$$my'' + by' + ky = 0, \quad y(a) = y_0, \quad y'(a) = y_1,$$

and $h(t)$ is the impulse response function.

7. USING VARIATION OF PARAMETERS TO FIND A PARTICULAR SOLUTION

A consequence of Theorem 6.1 is that a particular solution of

$$(7.1) \quad my'' + by' + ky = f(t)$$

is

$$(7.2) \quad y_p(t) = \int_a^t h(t-s)f(s) ds.$$

In this formula, $h(t)$ is the impulse response function, i.e., the solution of

$$my'' + by' + ky = 0, \quad y(0) = 0, \quad y'(0) = \frac{1}{m},$$

and a is any convenient number. This formula provides another way to find a particular solution of (7.1), in addition to undetermined coefficients.

Let's do an example from Chapter 4.6 this way.

Example. Find the general solution on $(-\frac{\pi}{2}, \frac{\pi}{2})$ to

$$\frac{d^2y}{dt^2} + y = \tan t.$$

Solution 1. The general solution of the homogeneous equation is

$$y = c_1 \cos t + c_2 \sin t.$$

A particular solution is given by formula (7.2) with $a = 0$. We already calculated this in formula (6.4):

$$y_p(t) = \sin t - \cos t \ln |\sec t + \tan t|.$$

However, $\sin t$ is a solution of the homogeneous equation, so if we subtract it, we will get a simpler particular solution:

$$y_p(t) = -\cos t \ln |\sec t + \tan t|.$$

Therefore the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln |\sec t + \tan t|.$$

Solution 2. The general solution of the homogeneous equation is

$$y = c_1 \cos t + c_2 \sin t.$$

A particular solution is given by formula (7.2) with $a = 0$. We calculated earlier that $h(t) = \sin t$, so

$$\begin{aligned} y_p(t) &= \int_0^t \sin(t-s) \tan s ds \\ &= \int_0^t (\sin t \cos s - \cos t \sin s) \tan s ds \\ &= \sin t \int_0^t \cos s \tan s ds - \cos t \int_0^t \sin s \tan s ds \\ &= \sin t \int_0^t \sin s ds - \cos t \int_0^t \frac{\sin^2 s}{\cos s} ds. \end{aligned}$$

Now

$$\int_0^t \sin s \, ds = -\cos s \Big|_0^t = -\cos t + \cos 0.$$

This must be multiplied by $\sin t$, which is a solution of the homogeneous equation. Therefore we ignore the term $\sin t \cos 0$ and just use $-\sin t \cos t$.

With the help of p. 196 in the text, we find that

$$\int_0^t \frac{\sin^2 s}{\cos s} \, ds = \ln |\sec s + \tan s| - \sin s \Big|_0^t = \ln |\sec t + \tan t| - \sin t - (\ln |\sec 0 + \tan 0| - \sin 0)$$

This must be multiplied by $\cos t$, which is a solution of the homogeneous equation. Therefore we ignore the term $\cos t(\ln |\sec 0 + \tan 0| - \sin 0)$ and just use $\cos t(\ln |\sec t + \tan t| - \sin t)$.

Therefore a particular solution is

$$\begin{aligned} y_p(t) &= -\sin t \cos t - \cos t(\ln |\sec t + \tan t| - \sin t) \\ &= -\cos t \ln |\sec t + \tan t|, \end{aligned}$$

and the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t - \cos t \ln |\sec t + \tan t|.$$

8. FROM THE IMPULSE RESPONSE FUNCTION TO LAPLACE TRANSFORMS

You walk into a lab and see a spring-mass system set up. You strike the mass with a hammer and measure the response $h(t)$. Now you know the impulse response function. Hence if the spring-mass system is started from rest and subject to *any time-dependent force* $f(t)$, you can predict the response! It will be

$$y(t) = \int_0^t h(t-s)f(s) \, ds.$$

That's pretty good, especially considering that you didn't bother to measure the mass m , the damping coefficient b , or the spring constant k . In effect, you are finding solutions of the differential equation without knowing what the differential equation is.

Let's be greedy and try to use $h(t)$, which we already know, to figure out what the differential operator $my'' + by' + ky$ is. Then we'll know m , b , and k without doing any tedious experiments to measure them.

I will describe one way to do this.

Recall how we used undetermined coefficients to find a particular to solution to the non-homogeneous equation

$$(8.1) \quad my'' + by' + ky = e^{st}.$$

(In this formula, s is a constant.) We assumed there was a solution of the form $y_p = Ae^{st}$ and substituted into (8.1):

$$(ms^2 + bs + k)Ae^{st} = e^{st}.$$

We concluded that $A = \frac{1}{ms^2 + bs + k}$. Therefore a particular solution of (8.1) is

$$y_p = \frac{1}{ms^2 + bs + k} e^{st}.$$

This works as long as s is not a root of the characteristic equation.

The function $H(s) = \frac{1}{ms^2+bs+k}$ is called the *transfer function* of the differential operator $my''+by'+ky$. If you knew the transfer function you could figure out the differential operator. Indeed, the characteristic polynomial of the differential operator is

$$ms^2 + bs + k = \frac{1}{H(s)},$$

and once you know the characteristic polynomial, you know the differential operator.

Can we use the stimulus response function $h(t)$ to find the transfer function $H(s)$? Let's give it a try.

Consider the differential equation (8.1) with rest initial conditions. The solution is

$$y(t) = \int_0^t h(t-r)e^{sr} dr.$$

(I have to use some letter other than s for the variable of integration. I chose to use r .) On the other hand, *any* solution of (8.1) has the form

$$y = c_1e^{r_1t} + c_2e^{r_2t} + H(s)e^{st},$$

where r_1 and r_2 are the roots of the characteristic polynomial. Hence there must be constants c_1 and c_2 such that

$$(8.2) \quad c_1e^{r_1t} + c_2e^{r_2t} + H(s)e^{st} = \int_0^t h(t-r)e^{sr} dr.$$

Now let's assume that s is bigger than both r_1 and r_2 . Multiply both sides of (8.2) by e^{-st} :

$$\begin{aligned} c_1e^{-(s-r_1)t} + c_2e^{-(s-r_2)t} + H(s) &= e^{-st} \int_0^t h(t-r)e^{sr} dr \\ &= \int_0^t h(t-r)e^{-s(t-r)} dr \\ &= \int_0^t h(u)e^{-su} du. \end{aligned}$$

The last simplification comes from the change of variables $u = t - r$, $du = -dr$.

Finally, let $t \rightarrow \infty$ on both sides of the previous equation:

$$H(s) = \int_0^\infty h(u)e^{-su} du.$$

This equation is usually written with t instead of u :

$$(8.3) \quad H(s) = \int_0^\infty h(t)e^{-st} dt.$$

We're done! Equation (8.3) tells us how to calculate the transfer function $H(s)$ if we know the impulse response function $h(t)$.

Equation (8.3) takes the function $h(t)$, defined for $t \geq 0$, and transforms it into a new function $H(s)$, defined for s greater than the larger of r_1 and r_2 . It says that $H(s)$ is the

Laplace transform of $h(t)$. We will learn more about Laplace transforms in Chapter 7 of our text.

MATHEMATICS DEPARTMENT, NORTH CAROLINA STATE UNIVERSITY, BOX 8205, RALEIGH, NC 27695
USA, 919-515-6533

E-mail address: `schechter@math.ncsu.edu`