DAFERMOS REGULARIZATION OF A DIFFUSIVE-DISPERSIVE EQUATION WITH CUBIC FLUX

STEPHEN SCHECTER AND MONIQUE RICHARDSON TAYLOR

Abstract. We study existence and spectral stability of stationary solutions of the Dafermos regularization of a much-studied diffusive-dispersive equation with cubic flux. Our study includes stationary solutions that corresponds to Riemann solutions consisting of an undercompressive shock wave followed by a compressive shock wave. We use geometric singular perturbation theory (1) to construct the solutions, and (2) to show that asmptotically, there are no large eigenvalues, and any order-one eigenvalues must be near -1 or a certain number λ^*. We give numerical evidence that λ^* is also -1. Finally, we use pseudoexponential dichotomies to show that in a space of exponentially decreasing functions, the essential spectrum is contained in $\operatorname{Re}\lambda \leq -\delta < 0$.

1. Introduction

Consider a system of viscous conservation laws in one space dimension, i.e., a partial differential equation of the form

$$u_T + f(u)_X = (B(u)u_X)_X, \tag{1.1}$$

with $X \in \mathbb{R}$, $T \in [0,\infty)$, $u \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$, and $B(u)$ is an $n \times n$ matrix for which all eigenvalues have positive real part. We impose constant boundary conditions

$$u(-\infty,T) = u^\ell, \quad u(+\infty,T) = u^r, \quad 0 \leq T < \infty, \tag{1.2}$$

and some initial condition $u(X,0) = u^0(X)$.

It is believed [1] that as $T \to \infty$, solutions of such intitial-boundary-value problems typically approach Riemann solutions for the system of conservation laws

$$u_T + f(u)_X = 0. \tag{1.3}$$

These are solutions of (1.3) that depend only on $x = \frac{X}{T}$, and that satisfy the boundary conditions

$$u(-\infty) = u^\ell, \quad u(+\infty) = u^r. \tag{1.4}$$

In numerical simulations, the convergence is seen when the solution is viewed in the rescaled spatial variable $x = \frac{X}{T}$; the rescaling counteracts the spreading of the solution as time increases. Discontinuities (shock waves) in the limiting Riemann solution satisfy the viscous profile criterion for the regularization (1.1), i.e., they correspond to traveling waves of (1.1). Speaking very roughly, Riemann solutions are believed to play the same role for (1.1)–(1.2) that equilibria play for ordinary differential equations: they are the simplest asymptotic states. An important difference, however, is that Riemann solutions are not solutions of (1.1) but only of the related equation (1.3).

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If the Riemann solution is a single shock wave, then it corresponds to a traveling wave solution of (1.1). Stability of such solutions has been studied by many authors, most recently using techniques developed by Zumbrun and collaborators [27, 6]. Since Riemann solutions other than a single shock wave do not correspond to explicit solutions of (1.1)–(1.2), the study of their stability is less advanced; see, however, [23] for Riemann solutions consisting of a single rarefaction, and [16] for Riemann solutions consisting of weak Lax shock waves.

Since it is in the variables (x, T) with $x = \frac{X}{T}$ that the convergence of solutions of (1.1)–(1.2) to Riemann solutions is observed, Lin and Schecter proposed in [15] to make the following change of variables in (1.1)–(1.2):

$$x = \frac{X}{T}, \quad t = \ln T. \tag{1.5}$$

(The substitution $t = \ln T$ is simply for convenience. Decay that is algebraic in T becomes exponential in t.) We obtain

$$u_t + (Df(u) - xI)u_x = e^{-t}(B(u)u_x)_x, \tag{1.6}$$

$$u(-\infty, t) = u^\ell, \quad u(+\infty, t) = u^r, \quad 0 \le t < \infty. \tag{1.7}$$

Of course the interval $0 \le t < \infty$ corresponds to $1 \le T < \infty$, but this is not important since we are interested in asymptotic behavior. The fact that (1.6) is nonautonomous implies that solutions can easily approach limits that are not themselves solutions.

In studying nonautonomous systems such as (1.6), it is natural to first freeze the time-varying coefficient and study the resulting autonomous system. In this case one sets $\epsilon = e^{-t}$; for large t, ϵ is small. One obtains

$$u_t + (Df(u) - xI)u_x = \epsilon(B(u)u_x)_x, \tag{1.8}$$

with the boundary conditions (1.7). Returning to (X, T) variables, (1.8) becomes

$$u_T + f(u)_X = \epsilon T(B(u)u_X)_X. \tag{1.9}$$

Equation (1.9) is the *Dafermos regularization* of the system of conservation laws (1.3) associated with the regularization (1.1) ([3]; see also [24, 25, 26]).

Stationary solutions of (1.8), (1.7) satisfy the ODE

$$(Df(u) - xI)u_x = \epsilon(B(u)u_x)_x, \tag{1.10}$$

with boundary conditions (1.4). We shall refer to a solution $\hat{u}^\epsilon(x)$ of (1.10), (1.4) as a *Riemann-Dafermos solution* of (1.8). It is known in many cases that near a Riemann solution of (1.3), with shock waves that satisfy the viscous profile criterion for $B(u)$, there is a Riemann-Dafermos solution $\hat{u}^\epsilon(x)$ of (1.8) [26, 19, 21, 17, 22]. In the latter four papers, the solution is constructed using geometric singular perturbation theory [9], an idea originally due to Szmolyan.

It is reasonable to expect that information about the stability of $\hat{u}^\epsilon(x)$ as a solution of (1.8) will be helpful in the study of the stability of the corresponding Riemann solution as an asymptotic state of (1.1).

In [15] Lin and Schecter studied the linearization of (1.9) at a Riemann-Dafermos solution $\hat{u}^\epsilon(x)$ for the case in which $B(u) \equiv I$, the underlying Riemann solution consists of exactly n compressive shock waves (also called classical or Lax shock waves), and the Riemann solution satisfies various nondegeneracy conditions. They found that, asymptotically as $\epsilon \to 0$, (1) a region of the form $\operatorname{Re} \lambda \ge -\delta$, $\delta > 0$, consists of resolvent points and eigenvalues, (2) large eigenvalues (of order $\frac{1}{\epsilon}$) correspond to eigenvalues of the linearization of (1.1) at a viscous profile for one of the shock waves, and (3) order one eigenvalues correspond to eigenvalues of the underlying Riemann solution as a solution of (1.3). In addition, in the limit $\epsilon = 0$, there is an eigenvalue -1 of multiplicity n that reflects the fact that each traveling wave can be shifted. For further work in this direction, see [13, 14, 20]. However, describing the spectrum for a fixed small ϵ rather than asymptotically remains an open problem.

In the present paper we consider, instead of (1.1), the diffusive-dispersive equation

$$u_T + (u^3)_X = \upsilon u_{XX} + \omega u_{XXX}, \tag{1.11}$$

with $\upsilon > 0$, $\omega > 0$, and boundary conditions (1.2). The rescaling $T \to \frac{1}{\sqrt{\omega}}T$, $X \to \frac{1}{\sqrt{\omega}}X$ converts (1.11) to

$$u_T + (u^3)_X = \alpha u_{XX} + u_{XXX}, \tag{1.12}$$

with $\alpha = \frac{\upsilon}{\sqrt{\omega}}$. This equation, sometimes called the modified Korteweg–deVries–Burgers equation, has attracted attention because the underlying conservation law

$$u_T + (u^3)_X = 0 \tag{1.13}$$

has shock waves that satisfy the viscous profile criterion for the regularization (1.12) but not for any regularization (1.11) with $\upsilon > 0$ and $\omega = 0$. Characteristics pass through these new shock waves; they are termed undercompressive. The paper [7] on (1.12) has inspired numerous subsequent studies; see the book [12].

Riemann problems for (1.13) can be solved using shock waves that satisfy the viscous profile criterion for the regularization (1.12). Some Riemann solutions consist of two shock waves with different speeds, one undercompressive and one compressive. Numerical simulations suggest that solutions of (1.12), (1.2), with appropriate u^ℓ and u^r, converge to these Riemann solutions as $T \to \infty$.

It therefore makes sense to consider the equation (1.12) from the point of view already sketched for equation (1.1). Higher-order equations have not previously been considered from this point of view. In addition, spectral stability of Riemann-Dafermos solutions corresponding to Riemann solutions that contain undercompressive shock waves has not been considered from this point of view even for equations of the form (1.1).

We make the change of variables (1.5) in (1.12), (1.2), and obtain the nonautonomous equation

$$u_t + (3u^2 - x)u_x = \alpha e^{-t} u_{xx} + e^{-2t} u_{xxx}, \tag{1.14}$$

with boundary conditions (1.7). Replacing e^{-t} by ϵ, we obtain

$$u_t + (3u^2 - x)u_x = \alpha \epsilon u_{xx} + \epsilon^2 u_{xxx}. \tag{1.15}$$

In (X, T) variables, (1.15) becomes

$$u_T + (u^3)_X = \alpha \epsilon T u_{XX} + (\epsilon T)^2 u_{XXX}, \tag{1.16}$$

and hence can be thought of as a sort of Dafermos regularization.

In Section 2 we review results from [7] about traveling waves for the diffusive-dispersive equation (1.12), and Riemann solutions of the conservation law (1.13) whose shock waves correspond to traveling waves of (1.12). We then show, using geometric singular perturbation theory, that corresponding to Riemann solutions that consist of a single compressive shock wave, or of an undercompressive shock wave followed by a compressive shock wave, there is, for small $\epsilon > 0$, a nearby Riemann-Dafermos solution of (1.15) with the same u^ℓ and u^r. In Section 3 we construct asymptotic expansions of various parts of the Riemann-Dafermos solutions.

In Section 4 we first review work of Dodd [4] that shows spectral stability of traveling waves of the diffusive-dispersive equation (1.12) that correspond to undercompressive shock waves. Dodd's work, together with work of Howard and Zumbrun [5], implies linear and nonlinear stability of these waves. Unfortunately it appears that stability of traveling waves of (1.12) that correspond to compressive shock waves has not been studied. We shall simply assume it.

We then linearize (1.12) at a Riemann-Dafermos solution and study eigenvalues of order $\frac{1}{\epsilon}$. We show that asymptotically, due to the spectral stability of the individual viscous profiles for the shock waves in the underlying Riemann solution, there are no such eigenvalues.

In Section 5 we study eigenvalues of order one. For a Riemann-Dafermos solution whose underlying Riemann solution is a single compressive shock wave, we show that asymptotically the only eigenvalue is $\lambda = -1$.

For a Riemann-Dafermos solution whose underlying Riemann solution consists of an undercompressive shock wave followed by a compressive shock wave, we show that asymptotically the only eigenvalues are -1 and a number λ^* for which we derive a formula. The -1 is associated with the compressive shock wave, and λ^* with the undercompressive shock wave. By analogy to earlier work, one expects λ^* to also be -1. We have not been able to show this. However, using Maple we have computed λ^* for one value of the parameters; to five decimal places, we obtained -1.00000.

The proofs for eigenvalues of order one are somewhat simpler than the corresponding treatment of order-one eigenvalues in [15], because the underlying conservation law is scalar. The assumption that the individual viscous profiles are spectrally stable is not needed. However, in the second case, the fact that one shock wave is undercompressive leads to a technical issue about exchange lemmas that we point out at the end of the section.

In Section 6 we show that for a small $\delta > 0$ and any fixed λ with $\operatorname{Re}\lambda > -\delta$, the resolvent equation can be solved in a space of exponentially decreasing functions for $\epsilon > 0$ sufficiently small. As in [15], the proof is based on pseudoexponential dichotomies.

2. Traveling waves, shock waves, Riemann solutions, and Riemann-Dafermos solutions

2.1. **Traveling waves.**

Traveling waves for (1.12) are solutions of the form $u(\eta)$, $\eta = X - sT$. They therefore satisfy the ODE

$$(3u^2 - s)u_\eta = \alpha u_{\eta\eta} + u_{\eta\eta\eta}. \tag{2.1}$$

A traveling wave with left state u^- and right state u^+ satisfies in addition the boundary conditions

$$u(-\infty) = u^-,\ u(\infty) = u^+,\ u_\eta(\pm\infty) = 0,\ \text{and}\ u_{\eta\eta}(\pm\infty) = 0. \tag{2.2}$$

Integrating (2.1) from $-\infty$ to η then yields

$$u^3 - su - \big((u^-)^3 - su^-\big) = \alpha u_\eta + u_{\eta\eta}. \tag{2.3}$$

Written as a system, (2.3) becomes the traveling wave system

$$u_\eta = v, \tag{2.4}$$

$$v_\eta = u^3 - su - \alpha v - z, \tag{2.5}$$

with

$$z = (u^-)^3 - su^-. \tag{2.6}$$

For fixed $\alpha > 0$, (2.4)–(2.5) is a 2-dimensional system parameterized by z and s. The point $(u^-, 0)$ is automatically an equilibrium of (2.4)–(2.5). In order that $(u^+, 0)$ also be an equilibrium, we require $z = (u^+)^3 - su^+$, so

$$s = \frac{(u^+)^3 - (u^-)^3}{u^+ - u^-}. \tag{2.7}$$

The system (2.4)–(2.5) has at most three equilibria. It has precisely three provided

$$4s^3 - 27z^2 > 0. \tag{2.8}$$

In this case, one equilibrium $(u, 0)$ has $3u^2 - s < 0$ and the other two have $3u^2 - s > 0$, one with $u > \left(\frac{s}{3}\right)^{\frac{1}{2}}$ and one with $u < -\left(\frac{s}{3}\right)^{\frac{1}{2}}$.

The linearization of (2.4)–(2.5) at an equilibrium $(u, 0)$ has the eigenvalues

$$\mu^{\pm}(u, s) = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 + 3u^2 - s}. \tag{2.9}$$

Since $\alpha > 0$, the equilibrium with $3u^2 - s < 0$ is an attractor, and the two with $3u^2 - s > 0$ are saddles.

Fix $u^- > 0$. Let $0 < s < 3(u^-)^2$, and let z be given by (2.6). Then $(u^-, 0)$ is a saddle for (2.4)–(2.5), and there are two other equilibria. According to [7], we have the following dichotomy.

Theorem 2.1. (1) *Suppose $0 < u^- < \frac{2}{3}\alpha\sqrt{2}$. Then (2.4)–(2.5) has a heteroclinic solution from $(u^-, 0)$ to a second equilibrium $(u^+, 0)$ if and only if $(u^+, 0)$ is the attractor.*

(2) *Suppose $u^- > \frac{2}{3}\alpha\sqrt{2}$. Let*

$$s(\alpha, u^-) = (u^-)^2 - \frac{\alpha\sqrt{2}}{3}u^- + \frac{2\alpha^2}{9}.$$

Then (2.4)–(2.5) has a heteroclinic solution from $(u^-, 0)$ to a second equilibrium $(u^+, 0)$ if and only if one of the following situations occurs.

(a) *$0 < s < s(\alpha, u^-)$ and $(u^+, 0)$ is the attractor.*

(b) *$s = s(\alpha, u^-)$ and $(u^+, 0)$ is the other saddle.*

In the second case, $u^+ = -u^- + \frac{\alpha\sqrt{2}}{3}$, and the connecting orbit from $(u^-, 0)$ to $(u^+, 0)$ is a portion of a parabola below the u-axis on which $u_\xi = v < 0$.

2.2. **Shock waves and rarefactions.** A shock wave for (1.13) is a weak solution of the form $u(x)$, $x = \frac{X}{T}$, with

$$u(x) = u^- \text{ for } x < s, \quad u(x) = u^+ \text{ for } x > s. \tag{2.10}$$

It is admissible for the regularization (1.12) if (1.12) has a traveling wave with velocity s from u^- to u^+. In particular, the Rankine-Hugoniot condition (2.7) is satisfied.

An admissible shock wave is compressive if any of the following equivalent conditions holds:

(1) $3(u^-)^2 > s > 3(u^+)^2$.

(2) In the XT-plane, characteristics enter the shock line $X = sT$ from both sides.

(3) The viscous profile is a saddle-to-attractor connection.

An admissible shock wave is undercompressive if any of the following equivalent conditions holds:

(1) $3(u^-)^2 > s$ and $3(u^+)^2 > s$.

(2) In the XT-plane, characteristics enter the shock line $X = sT$ from the left and leave it at the right.
(3) The viscous profile is a saddle-to-saddle connection.

A rarefaction for (1.13) is a smooth solution of the form $u(x)$, $x = \frac{X}{T}$. Rarefactions are obtained by solving the equation $3u^2 - x = 0$ for u.

2.3. **Riemann solutions.** A Riemann solution for (1.13) is a weak solution of the form $u(x)$, $x = \frac{X}{T}$, that satisfies boundary conditions of the form

$$u(-\infty) = u^\ell, \quad u(\infty) = u^r. \tag{2.11}$$

Riemann solutions are comprised of constant parts, rarefaction waves, and jump discontinuities. A jump discontinuity at $x = s$ with $\lim_{x\to s-} u(x) = u^-$ and $\lim_{x\to s+} u(x) = u^+$ is allowed if and only if (2.10) is an admissible shock wave. The discontinuity is itself termed a shock wave.

According to [7], we have the following result.

Theorem 2.2. *Let $\alpha > 0$.*

(1) *If $0 < u^\ell < \frac{2\alpha\sqrt{2}}{3}$ and $-\frac{u^\ell}{2} < u^r < u^\ell$, then the Riemann solution is a single compressive shock wave:*

$$u(x) = \begin{cases} u^\ell \text{ for } x < x^* \\ u^r \text{ for } x > x^* \end{cases},$$

(2) *If $u^\ell > \frac{2\alpha\sqrt{2}}{3}$ and $-u^\ell + \frac{\alpha\sqrt{2}}{3} < u^r < -\frac{u^\ell}{2}$, then the Riemann solution is an undercompressive shock wave with speed x^* followed by a compressive shock wave with speed $x^\diamond$, $x^* < x^\diamond$:*

$$u(x) = \begin{cases} u^\ell \text{ for } x < x^* \\ u^m \text{ for } x^* < x < x^\diamond \\ u^r \text{ for } x^\diamond < x \end{cases}.$$

2.4. **Riemann-Dafermos solution.** Riemann–Dafermos solutions are stationary solutions of (1.15). Hence they satisfy the equation

$$(3u^2 - x)u_x = \alpha\epsilon u_{xx} + \epsilon^2 u_{xxx}. \tag{2.12}$$

together with the boundary conditions (2.11). Equation (2.12) can be written as the following system:

$$\epsilon u_x = v, \tag{2.13}$$
$$\epsilon v_x = w, \tag{2.14}$$
$$\epsilon w_x = (3u^2 - x)v - \alpha w, \tag{2.15}$$
$$x_x = 1. \tag{2.16}$$

System (2.13)–(2.16) is the slow form of a slow-fast system. The change of variable $x = \epsilon\xi$ converts (2.13)–(2.16) into the fast form:

$$u_\xi = v, \tag{2.17}$$
$$v_\xi = w, \tag{2.18}$$
$$w_\xi = (3u^2 - x)v - \alpha w, \tag{2.19}$$
$$x_\xi = \epsilon. \tag{2.20}$$

Letting $\epsilon = 0$ in (2.17)–(2.20), we obtain the fast limit system

$$u_\xi = v, \tag{2.21}$$
$$v_\xi = w, \tag{2.22}$$
$$w_\xi = (3u^2 - x)v - \alpha w, \tag{2.23}$$
$$x_\xi = 0. \tag{2.24}$$

The set of equilibria of (2.21)–(2.24) is the ux-plane. The eigenvalues of the linearization of (2.21)–(2.24) at $(u, 0, 0, x)$ are 0, 0, and $\mu^\pm(u, x)$ given by (2.9) with $s = x$. For $3u^2 - x > 0$, $\mu^+(u, x) > 0$ and $\mu^-(u, x) < 0$; for $3u^2 - x < 0$, $\mu^\pm(u, x)$ both have negative real part.

In $uvwx$-space, for a small $\delta > 0$, let

$$\mathcal{P} = \{(u, 0, 0, x) : |u| \leq \frac{1}{\delta} \text{ and } -\infty < x \leq 3u^2 - \delta\},$$
$$\mathcal{P}^{u^*} = \{(u, 0, 0, x) : u = u^* \text{ and } -\infty < x \leq 3(u^*)^2 - \delta\} \subset \mathcal{P},$$
$$\mathcal{Q} = \{(u, 0, 0, x) : |u| \leq \frac{1}{\delta} \text{ and } 3u^2 + \delta \leq x < \infty\},$$
$$\mathcal{Q}^{u^*} = \{(u, 0, 0, x) : u = u^* \text{ and } 3(u^*)^2 + \delta \leq x < \infty\} \subset \mathcal{Q}.$$

See Figure 2.1. For the system (2.21)–(2.24), $\mathcal{P}$ and $\mathcal{Q}$ can be viewed as 2-dimensional

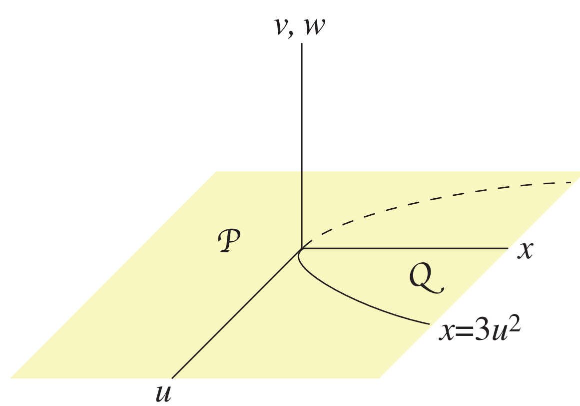

FIGURE 2.1. $uvwx$-space

normally hyperbolic manifolds of equilibria. (This is not obvious since $\mathcal{P}$ and $\mathcal{Q}$ are not compact. The argument requires a change of variables and compactification; see Appendix A.) Each point of $\mathcal{P}$ has a 1-dimensional unstable manifold and a 1-dimensional stable manifold; each point of $\mathcal{Q}$ has a 2-dimensional stable manifold. The ux-plane remains invariant under (2.17)–(2.20) for $\epsilon \neq 0$. It follows (see Appendix A) that for the system (2.17)–(2.20), for small ϵ, $\mathcal{P}$ and $\mathcal{Q}$ are 2-dimensional normally hyperbolic invariant manifolds. (By a common abuse of terminology, we will frequently use "invariant" to mean "locally invariant.") Each point of $\mathcal{P}$ has a 1-dimensional unstable fiber and a 1-dimensional stable fiber; each point of $\mathcal{Q}$ has a 2-dimensional stable fiber. Since $\mathcal{P}^{u^*}$ and $\mathcal{Q}^{u^*}$ remain invariant for $\epsilon \neq 0$, for small ϵ, $\mathcal{P}^{u^*}$ has 2-dimensional unstable and stable manifolds, denoted $W^u_\epsilon(\mathcal{P}^{u^*})$ and $W^s_\epsilon(\mathcal{P}^{u^*})$ respectively; and $\mathcal{Q}^{u^*}$ has a 3-dimensional stable manifold, denoted $W^s_\epsilon(\mathcal{Q}^{u^*})$.

Suppose, for a small $\epsilon > 0$, $W^u_\epsilon(\mathcal{P}^{u^\ell})$ and $W^s_\epsilon(\mathcal{Q}^{u^r})$ have nonempty intersection. Let $(u, v, w, x) = (u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi)$ be a solution in the intersection. Then $\hat{u}^\epsilon(x) = u^\epsilon(\frac{x}{\epsilon})$ is a Riemann-Dafermos solution.

Theorem 2.3. *Let $\alpha > 0$, and let (u^ℓ, u^r) satisfy the inequalities of Theorem 2.2 (1). Let x^* be the speed of the shock wave given by Theorem 2.2 (1). Let $u^*(\eta)$ be the viscous profile, i.e., the solution of (2.1) that satisfies $u^*(-\infty) = u^\ell$, $u^*(\infty) = u^r$, $u^*_\eta(\pm\infty) = u^*_{\eta\eta}(\pm\infty) = 0$. Let*

$$\Gamma_1 = \{(u^\ell, 0, 0, x) : -\infty < x \le x^*\},$$
$$\Gamma_2 = \{(u^*(\eta), v^*(\eta), w^*(\eta), x^*) : v^* = u^*_\eta,\ w^* = u^*_{\eta\eta},\ -\infty < \eta < \infty\},$$
$$\Gamma_3 = \{(u^r, 0, 0, x) : x^* \le x < \infty\}.$$

Then for small $\epsilon > 0$, $W^u_\epsilon(\mathcal{P}^{u^\ell})$, which has dimension 2, and $W^s_\epsilon(\mathcal{Q}^{u^r})$, which has dimension 3, intersect transversally in a curve

$$\Gamma_\epsilon = \{(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi) : -\infty < \xi < \infty\}.$$

As $\epsilon \to 0$, $\Gamma_\epsilon \to \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

See Figure 2.2 (a).

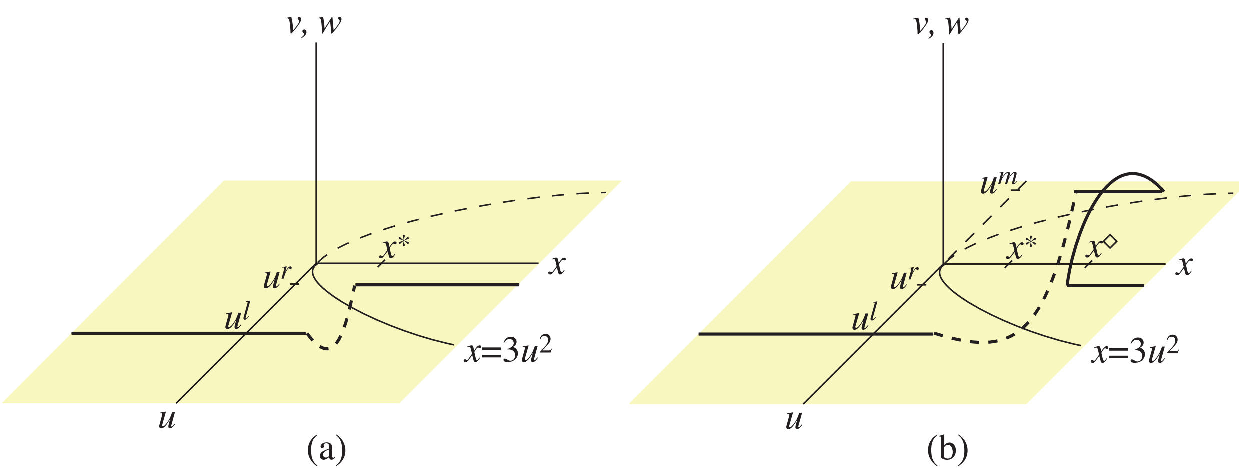

FIGURE 2.2. (a) The bold curve is $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ for Theorem 2.3. (b) The bold curve is $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$ for Theorem 2.4.

In order to study the intersection of $W^u_0(\mathcal{P}^{u^\ell})$ and $W^s_0(\mathcal{Q}^{u^r})$, it is convenient to replace the variable w in (2.21)–(2.24) by $z = u^3 - xu - \alpha v - w$. We obtain

$$u_\xi = v, \tag{2.25}$$
$$v_\xi = u^3 - xu - \alpha v - z, \tag{2.26}$$
$$z_\xi = 0, \tag{2.27}$$
$$x_\xi = 0. \tag{2.28}$$

This system can be viewed as a system of equations in uv-space parameterized by z and x; it is just the traveling wave system (2.4)–(2.5) with s replaced by x.

The sets $\mathcal{P}$, $\mathcal{P}^{u^\ell}$, and $\mathcal{Q}^{u^r}$ in $uvwx$-space correspond respectively to the following sets in in $uvzx$-space:

$$\begin{aligned}
\tilde{\mathcal{P}} &= \{(u,v,z,x) : |u| \le \frac{1}{\delta},\ v=0,\ z=u^3-xu,\ -\infty < x \le 3u^2-\delta\},\\
\tilde{\mathcal{P}}^{u^\ell} &= \{(u,v,z,x) : u=u^\ell,\ v=0,\ z=(u^\ell)^3-xu^\ell,\ -\infty < x \le 3(u^\ell)^2-\delta\},\\
\tilde{\mathcal{Q}}^{u^r} &= \{(u,v,z,x) : u=u^r,\ v=0,\ z=(u^r)^3-xu^r,\ 3(u^r)^2+\delta \le x < \infty\}.
\end{aligned}$$

Proof. We shall work in $uvzx$-coordinates. Let $z^* = (u^\ell)^3 - x^*u^\ell = (u^r)^3 - x^*u^r$. Then $W_0^u(\tilde{\mathcal{P}}^{u^\ell})$ and $W_0^s(\tilde{\mathcal{Q}}^{u^r})$ intersect along the curve $\tilde{\Gamma}_2 = \{(u,v,z,x) : u=u^*(\eta),\ v=v^*(\eta),\ z=z^*,\ x=x^*\}$.

We shall show that $W_0^u(\tilde{\mathcal{P}}^{u^\ell})$, which has dimension 2, and $W_0^s(\tilde{\mathcal{Q}}^{u^r})$, which has dimension 3, meet transversally along $\tilde{\Gamma}_2$. Then $W_0^u(\mathcal{P}^{u^\ell})$ and $W_0^s(\mathcal{Q}^{u^r})$ meet transversally along Γ_2. Hence for small ϵ, $W_\epsilon^u(\mathcal{P}^{u^\ell})$ and $W_\epsilon^s(\mathcal{Q}^{u^r})$ meet transversally near Γ_2, and the result follows.

There is a function h such that, near the point $(u^*(0), v^*(0), z^*, x^*)$ on this curve,

$$\begin{aligned}
W_0^u(\tilde{\mathcal{P}}) = \{(u,v,z,x) : u \in \text{ interval around } u^*(0),\ v = h(u,z,x),\\
z \in \text{ interval around } z^*,\ x \in \text{ interval around } x^*\}, \quad (2.29)
\end{aligned}$$

$$\begin{aligned}
W_0^u(\tilde{\mathcal{P}}^{u^\ell}) = \{(u,v,z,x) : u \in \text{ interval around } u^*(0),\ v = h(u,(u^\ell)^3-xu^\ell,x),\\
z = (u^\ell)^3 - xu^\ell,\ x \in \text{ interval around } x^*\}, \quad (2.30)
\end{aligned}$$

$$\begin{aligned}
W_0^s(\tilde{\mathcal{Q}}^{u^r}) = \{(u,v,z,x) : u \in \text{ interval around } u^*(0),\ v \in \text{ interval around } v^*(0),\\
z = (u^r)^3 - xu^r,\ x \in \text{ interval around } x^*\}. \quad (2.31)
\end{aligned}$$

Bases for the tangent spaces to $W_0^u(\tilde{\mathcal{P}}^{u^\ell})$ and $W_0^s(\tilde{\mathcal{Q}}^{u^r})$ at the point $(u^*(0), v^*(0), z^*, x^*)$ are

$$\left\{ \begin{pmatrix} 1 \\ h_u \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -u^\ell h_z + h_x \\ -u^\ell \\ 1 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -u^r \\ 1 \end{pmatrix} \right\} \quad (2.32)$$

respectively. These five vectors span $\mathbb{R}^4$ if and only if the second vector in the first set and the three vectors in the second set are linearly independent, which is the case if and only if $u^\ell \ne u^r$. In fact $u^r < u^\ell$ by the assumption of Theorem 2.2 (1). □

Theorem 2.4. *Let $\alpha > 0$, and let (u^ℓ, u^r) satisfy the inequalities of Theorem 2.2 (2). Let $x^* < x^\diamond$ be the speeds of the shock waves given by Theorem 2.2 (2). Let $u^*(\eta)$ and $u^\diamond(\eta)$ be the corresponding viscous profiles, which connect u^ℓ to u^m and u^m to u^r respectively. Let*

$$\begin{aligned}
\Gamma_1 &= \{(u^\ell,0,0,x) : -\infty < x \le x^*\},\\
\Gamma_2 &= \{(u^*(\eta), v^*(\eta), w^*(\eta), x^*) : v^* = u^*_\eta,\ w^* = u^*_{\eta\eta},\ -\infty < \eta < \infty\},\\
\Gamma_3 &= \{(u^m,0,0,x) : x^* \le x \le x^\diamond\},\\
\Gamma_4 &= \{(u^\diamond(\eta), v^\diamond(\eta), w^\diamond(\eta), x^\diamond) : v^\diamond = u^\diamond_\eta,\ w^\diamond = u^\diamond_{\eta\eta},\ -\infty < \eta < \infty\},\\
\Gamma_5 &= \{(u^r,0,0,x) : x^\diamond \le x < \infty\}.
\end{aligned}$$

Then for small $\epsilon > 0$, $W^u_\epsilon(\mathcal{P}^{u^\ell})$, *which has dimension 2, and* $W^s_\epsilon(\mathcal{Q}^{u^r})$, *which has dimension 3, intersect transversally in a curve*

$$\Gamma_\epsilon = \{(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi) : -\infty < \xi < \infty\}.$$

As $\epsilon \to 0$, $\Gamma_\epsilon \to \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$.

See Figure 2.2 (b).

Proof. We claim:

(1) $W^u_0(\mathcal{P}^{u^\ell})$, which has dimension 2, and $W^s_0(\mathcal{P})$, which has dimension 3, intersect transversally along Γ_2.
(2) $W^u_0(\mathcal{P}^{u^m})$, which has dimension 2, and $W^s_0(\mathcal{Q}^{u^r})$, which has dimension 3, intersect transversally along Γ_4.

The second claim is proved like Theorem 2.3. Once the first claim is proved, we note that by the simplest version of the exchange lemma ([10]), for small $\epsilon > 0$, $W^u_\epsilon(\mathcal{P}^{u^\ell})$, followed forward in time, arrives near the point $(u^m, 0, 0, x^\diamond)$ C^1-close to $W^u_\epsilon(\mathcal{P}^{u^m})$. Then the second claim implies that $W^u_\epsilon(\mathcal{P}^{u^\ell})$ and $W^s_\epsilon(\mathcal{Q}^{u^r})$ intersect transversally near Γ_4. The result follows.

To prove the first claim, we shall work in $uvzx$-coordinates. Let $z^* = (u^\ell)^3 - x^*u^\ell = (u^m)^3 - x^*u^m$. Then $W^u_0(\tilde{\mathcal{P}}^{u^\ell})$ and $W^s_0(\tilde{\mathcal{P}})$ intersect along the curve $\tilde{\Gamma}_2 = \{(u, v, z, x) : u = u^*(\eta),\ v = v^*(\eta),\ z = z^*,\ x = x^*\}$. There is a function h such that, near the point $(u^*(0), v^*(0), z^*, x^*)$ on this curve, $W^u_0(\tilde{\mathcal{P}})$ is given by (2.29), and $W^u_0(\tilde{\mathcal{P}}^{u^\ell})$ is given by (2.30). Similarly, there is a function k such that, near the same point,

$$W^s_0(\tilde{\mathcal{P}}) = \{(u, v, z, x) : u \in \text{ interval around } u^*(0),\ v = k(u, z, x),\\ z \in \text{ interval around } z^*,\ x \in \text{ interval around } x^*\}, \quad (2.33)$$

Bases for the tangent spaces to $W^u_0(\tilde{\mathcal{P}}^{u^\ell})$ and $W^s_0(\tilde{\mathcal{P}})$ at the point $(u^*(0), v^*(0), z^*, x^*)$ are

$$\left\{\begin{pmatrix}1\\h_u\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-u^\ell h_z + h_x\\-u^\ell\\1\end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix}1\\k_u\\0\\0\end{pmatrix}, \begin{pmatrix}0\\k_z\\1\\0\end{pmatrix}, \begin{pmatrix}0\\k_x\\0\\1\end{pmatrix}\right\} \quad (2.34)$$

respectively. Since $W^u_0(\tilde{\mathcal{P}}^{u^\ell})$ and $W^s_0(\tilde{\mathcal{P}})$ intersect along $\tilde{\Gamma}_2$, $h(u, z^*, x^*) = k(u, z^*, x^*)$ for all u, so the first vector in the first set and the first vector in the second set are the same. Therefore these five vectors span $\mathbb{R}^4$ if and only if the second vector in the first set and the three vectors in the second set are linearly independent, which is the case if and only if $-u^\ell h_z + h_x \neq -u^\ell k_z + k_x$ at $(u^*(0), z^*, x^*)$.

Consider the system (2.25)–(2.25) with $z = (u^\ell)^3 - xu^\ell$, a 2-dimensional system parameterized by x, which we consider for x near x^*:

$$u_\xi = v, \quad (2.35)$$
$$v_\xi = u^3 - (u^\ell)^3 - x(u - u^\ell) - \alpha v, \quad (2.36)$$

We can measure the distance between the unstable manifold of $(u^\ell, 0)$ and the stable manifold of the saddle near $(u^m, 0)$ by the difference between their v-coordinates on the line $u = u^*(0)$. This difference is given by

$$S(x) = h(u^*(0), (u^\ell)^3 - xu^\ell, x) - k(u^*(0), (u^\ell)^3 - xu^\ell, x).$$

We have $S(x^*) = 0$ and $S' = -u^\ell h_z + h_x - (-u^\ell k_z + k_x)$, evaluated at $(u^*(0), (u^\ell)^3 - xu^\ell, x)$. Thus our four vectors are linearly independent if and only if $S'(x^*) \neq 0$.

The linearization of (2.35)–(2.36), with $x = x^*$, along $(u^*(\xi), v^*(\xi))$ is

$$X_\xi = A(\xi)X, \quad A(\xi) = \begin{pmatrix} 0 & 1 \\ 3u^*(\xi)^2 - x^* & -\alpha \end{pmatrix}. \tag{2.37}$$

Let

$$\psi^2(\xi) = \begin{pmatrix} -e^{\alpha\xi}u^*_{\xi\xi} & e^{\alpha\xi}u^*_\xi \end{pmatrix}. \tag{2.38}$$

Up to a constant multiple, $\psi^2(\xi)$ is the unique bounded solution of the adjoint equation of (2.37), i.e., of $\psi_\xi = -\psi A(\xi)$.

Up to a constant multiple, $S'(x^*)$ is given by the Melnikov integral

$$S'(x^*) = \int_{-\infty}^{\infty} \psi^2(\xi) \begin{pmatrix} 0 \\ u^\ell - u^*(\xi) \end{pmatrix} d\xi = \int_{-\infty}^{\infty} e^{\alpha\xi} u^*_\xi (u^\ell - u^*(\xi))\, d\xi.$$

(The column vector is the partial derivative of the right hand side of (2.35)–(2.36) with respect to x at $(u, v, x) = (u^*(\xi), v^*(\xi), x^*)$.) From Theorem 2.1 (2), the connecting orbit from $(u^\ell, 0)$ to $(u^m, 0)$ has $u^*_\xi < 0$ and $u^\ell - u^*(\xi) > 0$ for all ξ. It follows that $S'(x^*) \neq 0$. □

3. Asymptotic expansion of connecting solutions

Consider a smooth family of solutions of (2.17)–(2.20), $(u(\xi, \epsilon), v(\xi, \epsilon), w(\xi, \epsilon), x(\xi, \epsilon))$, $0 \leq \epsilon < \epsilon_0$, that are asymptotic to $\mathcal{P}$ or $\mathcal{Q}$ at both ends. From the theory of normally hyperbolic invariant manifolds, they are asymptotic to solutions in $\mathcal{P}$ or $\mathcal{Q}$, which must take the form

$$\left(u^\pm(\xi, \epsilon), v^\pm(\xi, \epsilon), w^\pm(\xi, \epsilon), x^\pm(\xi, \epsilon)\right) = \left(\sum_{i=0}^{\infty} \epsilon^i u_i^\pm, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i^\pm \right), \tag{3.1}$$

with $u_i^\pm$ and $x_i^\pm$ constants. Which of these constants are given and which must be determined will depend on the situation that we consider. We write

$$u(\xi, \epsilon) = \sum_{i=0}^{\infty} \epsilon^i u_i(\xi), \quad x(\xi, \epsilon) = \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i, \tag{3.2}$$

where the x_i are constants. We see immediately that $x_i = x_i^- = x_i^+$ for all i. We rewrite (2.17)–(2.20) as

$$u_{\xi\xi\xi} = (3u^2 - x)u_\xi - \alpha u_{\xi\xi}, \quad x_\xi = \epsilon. \tag{3.3}$$

Substituting (3.2) into the first equation of (3.3), we obtain

$$\sum_{i=0}^{\infty} \epsilon^i u_{i\,\xi\xi\xi} = \left(3 \left(\sum_{i=0}^{\infty} \epsilon^i u_i \right)^2 - \epsilon\xi - \sum_{i=0}^{\infty} \epsilon^i x_i \right) \sum_{i=0}^{\infty} \epsilon^i u_{i\,\xi} - \alpha \sum_{i=0}^{\infty} \epsilon^i u_{i\,\xi\xi}. \tag{3.4}$$

Equating terms with the same powers of ϵ, and using the fact that our solution is asymptotic as $\xi \to \pm\infty$ to the solutions (3.1), we obtain, for ϵ^0 and ϵ^1, the equations

$$u_{0\,\xi\xi\xi} = (3u_0^2 - x_0)u_{0\,\xi} - \alpha u_{0\,\xi\xi}, \quad u_0(\pm\infty) = u_0^\pm, \quad u_{0\,\xi}(\pm\infty) = u_{0\,\xi\xi}(\pm\infty) = 0, \tag{3.5}$$

$$\begin{aligned} u_{1\,\xi\xi\xi} &= \left((3u_0^2 - x_0)u_1\right)_\xi - (x_1 + \xi)u_{0\,\xi} - \alpha u_{1\,\xi\xi}, \\ &\quad u_1(\pm\infty) = u_1^\pm, \quad u_{1\,\xi}(\pm\infty) = u_{1\,\xi\xi}(\pm\infty) = 0. \end{aligned} \tag{3.6}$$

For $k \geq 2$ we obtain

$$u_{k\,\xi\xi\xi} = \big((3u_0^2 - x_0)u_k\big)_\xi - x_k u_{0\,\xi} + P_k(u_0, \dots, u_{k-1}, u_{0\,\xi}, \dots, u_{k-1\,\xi}, x_0, \dots, x_{k-1}, \xi) - \alpha u_{k\,\xi\xi},$$
$$u_k(\pm\infty) = u_k^\pm, \quad u_{k\,\xi}(\pm\infty) = u_{k\,\xi\xi}(\pm\infty) = 0. \quad (3.7)$$

Integrating (3.5) from $-\infty$ to ξ, we obtain

$$u_{0\,\xi\xi} = u_0(\xi)^3 - x_0 u_0(\xi) - ((u_0^-)^3 - x_0 u_0^-) - \alpha u_{0\,\xi}. \quad (3.8)$$

Compare (2.3) and (2.4)–(2.5). We assume:

(C) x_0 and $u_0^\pm$ have been chosen so that the traveling wave equation (2.4)–(2.5), with $s = x_0$ and $z = (u_0^-)^3 - x_0 u_0^- = (u_0^+)^3 - x_0 u_0^+$, has a solution $(u_0(\xi), u_{0\,\xi}(\xi))$ that approaches $(u_0^\pm, 0)$ exponentially as $\xi \to \pm\infty$.

This choice of x_0, $u_0^\pm$, and $u_0(\xi)$ satisfies (3.5), and the assumption that $u_{0\,\xi}(\xi) \to 0$ exponentially as $\xi \to \pm\infty$ justifies the integration we did. Also, it implies that all derivatives of $u_0(\xi)$ approach 0 exponentially as $\xi \to \pm\infty$, which justifies further integrations.

Integrating (3.6) from $-\infty$ to ξ, we obtain

$$u_{1\,\xi\xi} = (3u_0(\xi)^2 - x_0)u_1(\xi) - (3(u_0^-)^2 - x_0)u_1^- - x_1(u_0(\xi) - u_0^-) - \int_{-\infty}^{\xi} \zeta u_{0\,\zeta}\, d\zeta - \alpha u_{1\,\xi}. \quad (3.9)$$

Setting $\xi = \infty$ in (3.9) yields

$$0 = (3(u_0^+)^2 - x_0)u_1^+ - (3(u_0^-)^2 - x_0)u_1^- - x_1(u_0^+ - u_0^-) - \int_{-\infty}^{\infty} \xi u_{0\,\xi}\, d\xi. \quad (3.10)$$

Further insight into (3.9) may be obtained by writing it as an inhomogeneous linear system:

$$X_\xi = A(\xi)X + H(\xi), \quad X(\xi) = \begin{pmatrix} u_1(\xi) \\ v_1(\xi) \end{pmatrix}, \quad A(\xi) = \begin{pmatrix} 0 & 1 \\ 3u_0(\xi)^2 - x_0 & -\alpha \end{pmatrix},$$
$$H(\xi) = \begin{pmatrix} 0 \\ h(\xi) \end{pmatrix}, \quad h(\xi) = -(3(u_0^-)^2 - x_0)u_1^- - x_1(u_0(\xi) - u_0^-) - \int_{-\infty}^{\xi} \zeta u_{0\,\zeta}\, d\zeta. \quad (3.11)$$

Compare the formula for $A(\xi)$ to (2.37); $X_\xi = A(\xi)X$ is also the linearization of the traveling wave system (2.4)–(2.5) along the solution $(u_0(\xi), u_{0\,\xi}(\xi))$. Let

$$X^1(\xi) = \begin{pmatrix} u^*_\xi \\ u^*_{\xi\xi} \end{pmatrix}, \quad \psi^2(\xi) = \begin{pmatrix} e^{\alpha\xi} u^*_{\xi\xi} & -e^{\alpha\xi} u^*_\xi \end{pmatrix}$$

Compare the formula for $\psi^2(\xi)$ to (2.38). Up to a constant multiple, the only bounded solution of $X_\xi = A(\xi)X$ is $X^1(\xi)$, and the only bounded solution of the adjoint equation $Y = -YA(\xi)$ is $\psi^2(\xi)$.

Let $X^2(\xi)$ be a solution of (2.37) that is linearly independent of $X^1(\xi)$, let $\mathbf{X}(\xi) = \begin{pmatrix} X^1(\xi) & X^2(\xi) \end{pmatrix}$. Then the second row of $\mathbf{X^{-1}}(\xi)$ is a multiple of $\psi^2(\xi)$; we assume $X^2(\xi)$ is chosen so that it is precisely $\psi^2(\xi)$. We then define $\psi^1(\xi)$ to be the first row of $\mathbf{X^{-1}}(\xi)$:

$$\mathbf{X^{-1}}(\xi) = \begin{pmatrix} \psi^1(\xi) \\ \psi^2(\xi) \end{pmatrix}.$$

Note that $h(\xi)$ has finite limits at $\xi = \pm\infty$.

Proposition 3.1. *Suppose that (C) holds, and for (2.4)–(2.5), with $s = x_0$ and $z = (u_0^-)^3 - x_0u_0^- = (u_0^+)^3 - x_0u_0^+$, $(u_0^-, 0)$ is a saddle and $(u_0^+, 0)$ is an attractor. Let x_1 and $u_1^\pm$ be given. Then the equation (3.9) has a solution $u_1(\xi)$ such that $u_1(\pm\infty) = u_1^\pm$, $u_{1\xi}(\pm\infty) = 0$, and $(u_1(\xi), u_{1\xi}(\xi))$ converges exponentially as $\xi \to \pm\infty$, if and only if the triple $(x_1, u_1^\pm)$ satisfies (3.10). The function $u_1(\xi)$ is unique up to addition of a multiple of $u_{0\xi}$.*

Proof. The derivation of (3.10) shows its necessity; we will show its sufficiency. Assume that $(x_1, u_1^\pm)$ satisfies (3.10). Note that x_1 and u_1^- are used in the definition of H in (3.11) but u_1^+ is not. The desired solution exists if and only if $X_\xi = A(\xi)X + H(\xi)$ has a solution $X(\xi)$ such that $X(\pm\infty) = (u_1^\pm, 0)$ and $X(\xi)$ converges exponentially as $\xi \to \pm\infty$.

On $-\infty < \xi \le 0$, $X_\xi = A(\xi)X$ has an exponential dichotomy with 1-dimensional unstable space spanned by $X^1(\xi)$ and 1-dimensional stable space, which we may take to be the span of $X^2(\xi)$. Then

$$P_-(\xi) = X^2(\xi)\psi^2(\xi), \quad I - P_-(\xi) = X^1(\xi)\psi^1(\xi), \quad -\infty < \xi \le 0, \tag{3.12}$$

are projections onto the stable and unstable spaces respectively.

On $0 \le \xi < \infty$ there is an exponential dichotomy with 2-dimensional stable space.

Let $\Phi(\xi,\zeta) = \mathbf{X}(\xi)\mathbf{X}^{-1}(\zeta)$ be the family of state transition matrices for $X_\xi = A(\xi)X$. If $X(\xi)$ is a bounded solution of $X_\xi = A(\xi)X + H(\xi)$, then for $-\infty < \xi \le 0$, $P_-(\xi)X(\xi)$ is uniquely defined by

$$\begin{aligned} P_-(\xi)X(\xi) &= \int_{-\infty}^{\xi} \Phi(\xi,\zeta)P_-(\zeta)H(\zeta)\,d\zeta \\ &= \int_{-\infty}^{\xi} \Phi(\xi,\zeta)X^2(\zeta)\psi^2(\zeta)H(\zeta)\,d\zeta = X^2(\xi)\int_{-\infty}^{\xi}\psi^2(\zeta)H(\zeta)\,d\zeta. \end{aligned} \tag{3.13}$$

This equation defines $P_-(0)X(0)$. We shall take $(I - P_-(0))X(0)$ to be 0, so that $X(0) = P_-(0)X(0)$, a multiple of $X^2(0)$. Then on $-\infty < \xi \le 0$,

$$\begin{aligned} (I - P_-(\xi))X(\xi) &= \int_0^{\xi} \Phi(\xi,\zeta)(I - P_-(\zeta))H(\zeta)\,d\zeta \\ &= \int_0^{\xi} \Phi(\xi,\zeta)X^1(\zeta)\psi^1(\zeta)H(\zeta)\,d\zeta = X^1(\xi)\int_0^{\xi}\psi^1(\zeta)H(\zeta)\,d\zeta. \end{aligned} \tag{3.14}$$

Finally, on $0 \le \xi < \infty$,

$$X(\xi) = \Phi(\xi,0)X(0) + \int_0^{\xi} \Phi(\xi,\zeta)H(\zeta)\,d\zeta.$$

One easily checks that $X(\xi)$ is a bounded solution of $X_\xi = A(\xi)X + H(\xi)$. All other bounded solutions of $X_\xi = A(\xi)X + H(\xi)$ are obtained by adding bounded solutions of $X_\xi = A(\xi)X$, i.e., multiples of $X^1(\xi)$.

From the fact that $X(\xi)$ is bounded, one can show that X_ξ and $X_{\xi\xi}$ approach 0 exponentially as $\xi \to \pm\infty$. Therefore $X(\xi)$ approaches limits exponentially as $\xi \to \pm\infty$. Let $u_1(\xi)$ be the first component of $X(\xi)$, so that $u_1(\xi)$ is a solution of (3.9). Letting $\xi \to -\infty$ in (3.9), we see that $u_1(-\infty) = u_1^-$. Letting $\xi \to \infty$ in (3.9), we see that $u_1(\infty) = \bar{u}_1^+$ exists and satisfies (3.10) with u_1^+ replaced by $\bar{u}_1^+$. It follows that $\bar{u}_1^+ = u_1^+$. ☐

Proposition 3.2. *Suppose that (C) holds, and for (2.4)–(2.5), with $s = x_0$ and $z = (u_0^-)^3 - x_0u_0^- = (u_0^+)^3 - x_0u_0^+$, $(u_0^-, 0)$ and $(u_0^+, 0)$ are both saddles. Let x_1 and $u_1^\pm$ be given. Then the* *equation* (3.9) *has a solution* $u_1(\xi)$ *such that* $u_1(\pm\infty) = u_1^\pm$, $u_{1\xi}(\pm\infty) = 0$, *and* $(u(\xi), u_{1\xi}(\xi))$ *converges exponentially as* $\xi \to \pm\infty$, *if and only if the triple* $(x_1, u_1^\pm)$ *satisfies* (3.10), *and in addition*

$$\int_{-\infty}^{\infty} \psi^2(\xi) H(\xi)\, d\xi = \int_{-\infty}^{\infty} e^{\alpha\xi} u_{0\xi} h(\xi)\, d\xi = 0. \tag{3.15}$$

The function $u_1(\xi)$ *is unique up to addition of a multiple of* $u_{0\xi}$.

Proof. As in the proof of Proposition 3.1, we assume that $(x_1, u_1^\pm)$ satisfies (3.10), construct a solution $X(\xi)$ of $X_\xi = A(\xi)X + H(\xi)$ such that $X(\pm\infty) = (u_1^\pm, 0)$, and let $u_1(\xi)$ be the first component of $X(\xi)$. We omit some details that are covered in the previous proof.

On both $-\infty < \xi \le 0$ and $0 \le \xi < \infty$, $X_\xi = A(\xi)X$ has exponential dichotomies with 1-dimensional unstable space and 1-dimensional stable space. We take the stable space on $-\infty < \xi \le 0$ and the unstable space on $0 \le \xi < \infty$ to be the span of $X^2(\xi)$. Let $P_\pm(\xi)$ be associated families of projections onto the stable spaces. On $-\infty < \xi \le 0$, the projections are given by (3.12). On $0 \le \xi < \infty$, they are given by

$$P_+(\xi) = X^1(\xi)\psi^1(\xi), \quad I - P_+(\xi) = X^2(\xi)\psi^2(\xi), \quad -\infty < \xi \le 0. \tag{3.16}$$

If $X(\xi)$ is a bounded solution of $X_\xi = A(\xi)X + H(\xi)$, then for $-\infty < \xi \le 0$, $P_-(\xi)X(\xi)$ is uniquely defined by (3.13), and for $0 \le \xi < \infty$, $(I - P_+(\xi))X(\xi)$ is uniquely defined by

$$\begin{aligned}(I - P_+(\xi))X(\xi) &= \int_{\infty}^{\xi} \Phi(\xi,\zeta)(I - P_+(\zeta))H(\zeta)\, d\zeta \\ &= \int_{\infty}^{\xi} \Phi(\xi,\zeta)X^2(\zeta)\psi^2(\zeta)H(\zeta)\, d\zeta = X^2(\xi)\int_{\infty}^{\xi} \psi^2(\zeta)H(\zeta)\, d\zeta. \end{aligned} \tag{3.17}$$

In order that the X^2-components of the two halves of the solution agree at $\xi = 0$, we must have (3.15).

If we take the X^1-component of the solution to be 0 at $\xi = 0$, then on $-\infty < \xi < 0$, the X^1-component of the solution is given by (3.14), and on $0 \le \xi < \infty$ by

$$\begin{aligned}P_+(\xi))X(\xi) &= \int_{0}^{\xi} \Phi(\xi,\zeta)P_+(\zeta)H(\zeta)\, d\zeta \\ &= \int_{0}^{\xi} \Phi(\xi,\zeta)X^1(\zeta)\psi^1(\zeta)H(\zeta)\, d\zeta = X^1(\xi)\int_{0}^{\xi} \psi^1(\zeta)H(\zeta)\, d\zeta. \end{aligned} \tag{3.18}$$

□

One can show inductively that each $u_k(\xi)$ approaches constants $u_k^\pm$ exponentially as $\xi \to \pm\infty$, and that propositions analogous to Propositions 3.1 and 3.2 hold for each k.

We note that the choice of $u_0(\xi)$ is not unique; it can be time-shifted. The choice of $u_0(\xi)$ determines the integral in (3.10), and hence determines the triples $(x_1, u_1^\pm)$ that can be used in Propositions 3.1 and 3.2. The choice of $u_1(\xi)$ in these propositions is again not unique. It will determine H_2, and hence the triples $(x_2, u_2^\pm)$ that can be used at the next level; etc.

3.1. **One-wave Riemann solution.** In the situation of Theorem 2.3, the transversal intersection of $W^u(\mathcal{P}^{u^\ell})$ and $W^s(\mathcal{Q}^{u^r})$ consists of one solution for each ϵ. Its u component can be expanded as (3.2) with $u_0(\xi) = u^*(\xi)$ and $x_0 = x^*$. The solutions (3.1) in $\mathcal{P}^{u^\ell}$ and $\mathcal{Q}^{u^r}$ to which these solutions are asymptotic must take the form

$$\left(u^\ell, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right) \text{ and } \left(u^r, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right)$$

with $x_0 = x^*$ and the other x_i's to be determined. In particular, $u_0^- = u^\ell$, $u_0^+ = u^r$, and $u_i^\pm = 0$ for $i \geq 1$. Equation (3.10) then yields

$$x_1 = \frac{\int_{-\infty}^{\infty} \xi u_\xi^* \, d\xi}{u^\ell - u^r}.$$

We can find $u_1(\xi)$ using the proof of Proposition 3.1.

3.2. **Two-wave Riemann solution: first wave.** In the situation of Theorem 2.4, the transversal intersection of $W^u(\mathcal{P}^{u^\ell})$ and $W^s(\mathcal{P})$ consists of one solution for each ϵ. Its u component can be expanded as (3.2) with $u_0(\xi) = u^*(\xi)$ and $x_0 = x^*$. The solutions (3.1) in $\mathcal{P}^{u^\ell}$ and $\mathcal{Q}^{u^r}$ to which these solutions are asymptotic must take the form

$$\left(u^\ell, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right) \text{ and } \left(\sum_{i=0}^{\infty} \epsilon^i u_i^m, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right)$$

with $x_0 = x^*$, $u_0^m = u^m$, and the other u_i^m's and x_i's to be determined. In particular, $u_0^- = u^\ell$ and $u_i^- = 0$ for $i \geq 1$.

From (3.10), (3.11), and (3.15) we have

$$0 = (3(u^m)^2 - x^*)u_1^m - x_1(u^m - u^\ell) - \int_{-\infty}^{\infty} \xi u_\xi^* \, d\xi, \tag{3.19}$$

$$0 = x_1 \int_{-\infty}^{\infty} e^{\alpha\xi}(u^*(\xi) - u^\ell)u_\xi^* \, d\xi + \int_{-\infty}^{\infty} e^{\alpha\xi} u_\xi^* \int_{-\infty}^{\xi} \zeta u_\zeta^* \, d\zeta \, d\xi. \tag{3.20}$$

These two equations determine x_1 and u_1^m. We can find $u_1(\xi)$ using the proof of Proposition 3.2.

3.3. **Two-wave Riemann solution: second wave.** In the situation of Theorem 2.4, the transversal intersection of $W^u(\mathcal{P})$ and $W^s(\mathcal{Q}^{u^r})$, both of which have dimension 3, consists of a 2-dimensional surface of solutions for each ϵ. The u component of the solution can be expanded as (3.2) with $u_0(\xi) = u^\diamond(\xi)$ and $x_0 = x^\diamond$. The solutions (3.1) in $\mathcal{P}$ and $\mathcal{Q}^{u^r}$ to which these solutions are asymptotic must take the form

$$\left(\sum_{i=0}^{\infty} \epsilon^i u_i^m, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right) \text{ and } \left(u^r, 0, 0, \epsilon\xi + \sum_{i=0}^{\infty} \epsilon^i x_i\right)$$

with $x_0 = x^\diamond$, all u_i^m's equal to their values determined in subsection 3.2 (so that the two parts of the solution will match), and the remaining x_i's to be determined. From (3.10) we have

$$0 = -(3(u^m)^2 - x^\diamond)u_1^m - x_1(u^r - u^m) - \int_{-\infty}^{\infty} \xi u_\xi^\diamond \, d\xi, \tag{3.21}$$

which determines x_1. We can find $u_1(\xi)$ using the proof of Proposition 3.1.

4. Linearization and Large Eigenvalues

4.1. **Linearized stability of traveling waves.** Before beginning our study of the spectral stability of Riemann-Dafermos solutions, we review what is known about spectral stability of the traveling waves of Section 2.1. In (1.12) we replace X by $\eta = X - sT$ and obtain

$$u_T + (3u^2 - sI)u_\eta = \alpha u_{\eta\eta} + u_{\eta\eta\eta}. \tag{4.1}$$

A traveling wave $u(\eta)$, $\eta = X - sT$, for (1.12) is an equilibrium of (4.1). Linearizing (4.1) at such a traveling wave, we obtain

$$U_T + (3u(\eta)^2 - sI)U_\eta + 6u(\eta)u_\eta U = \alpha U_{\eta\eta} + U_{\eta\eta\eta}. \tag{4.2}$$

The eigenvalue equation is

$$\rho U + (3u(\eta)^2 - sI)U_\eta + 6u(\eta)u_\eta U = \alpha U_{\eta\eta} + U_{\eta\eta\eta}. \tag{4.3}$$

Dodd [4] has shown

Theorem 4.1. *For each $u^- > 0$, there exists $\alpha_0 > 0$ such that the following is true. Let $0 < \alpha < \alpha_0$, let $s = s(\alpha, u^-)$ be the speed given by Theorem 2.1 (2), and let $(u(\eta), v(\eta))$ be the saddle-to-saddle connection of (2.4)–(2.5) from u^- to $u^+ = -u^- + \frac{\alpha\sqrt{2}}{3}$. Then the eigenvalue equation (4.3), with $\operatorname{Re}\rho \geq 0$, has no nontrivial solutions in L^2, except that $\rho = 0$ has a 1-dimensional space of solutions spanned by u_η. Moreover, $\rho = 0$ is a simple zero of the corresponding Evans function.*

For background on the Evans function, see [18].

A traveling wave solution of (1.12) whose linearization satisfies the conclusions of Theorem 4.1 is called spectrally stable. The spectral stability of traveling waves that correspond to saddle-to-attractor connections of (2.4)–(2.5) does not seem to have been studied. Whenever necessary, we shall simply assume that the viscous profiles of shock waves that occur in the Riemann solutions in which we are interested are spectrally stable.

4.2. **Linearization at a Riemann-Dafermos solution.** In the situation of Theorem 2.3 or 2.4, let $(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi)$ be the solution of (2.17)–(2.18) given by the theorem, so that $\hat{u}^\epsilon(x) = u^\epsilon(\frac{\xi}{\epsilon})$ is a Riemann–Dafermos solution. We linearize (1.15) at $\hat{u}^\epsilon(x)$ and obtain

$$\hat{U}_t + (3(\hat{u}^\epsilon)^2 - x)\hat{U}_x + 6\hat{u}^\epsilon(\hat{u}^\epsilon)'\hat{U} = \alpha\epsilon\hat{U}_{xx} + \epsilon^2\hat{U}_{xxx}. \tag{4.4}$$

We look for solutions of the form $\hat{U}(x,t) = e^{\lambda t}U(x)$. Substituting into (4.4) yields

$$\lambda U + (3(\hat{u}^\epsilon)^2 - x)U_x + 6\hat{u}^\epsilon(\hat{u}^\epsilon)'U \;=\; \alpha\epsilon U_{xx} + \epsilon^2 U_{xxx}. \tag{4.5}$$

The corresponding system is

$$\begin{aligned} \epsilon U_x &= V, \\ \epsilon V_x &= W, \\ \epsilon W_x &= \epsilon\lambda U + (3(\hat{u}^\epsilon)^2 - x)V + 6\epsilon\hat{u}^\epsilon(\hat{u}^\epsilon)'U - \alpha W. \end{aligned}$$

The substitution $x = \epsilon\xi$ yields

$$U_\xi = V, \tag{4.6}$$

$$V_\xi = W, \tag{4.7}$$

$$W_\xi = \epsilon\lambda U + (3(u^\epsilon)^2 - x)V + 6u^\epsilon u^\epsilon_\xi U - \alpha W. \tag{4.8}$$

Let $\rho = \epsilon\lambda$. We can merge together the linear system with the fast system (2.17)–(2.20) to obtain the following 7-dimensional system, in which ρ is a parameter:

$$u_\xi = v, \tag{4.9}$$
$$v_\xi = w, \tag{4.10}$$
$$w_\xi = (3u^2 - x)v - \alpha w, \tag{4.11}$$
$$x_\xi = \epsilon, \tag{4.12}$$
$$U_\xi = V, \tag{4.13}$$
$$V_\xi = W, \tag{4.14}$$
$$W_\xi = \rho U + (3u^2 - x)V + 6uvU - \alpha W. \tag{4.15}$$

Note that (4.13)–(4.15), with $(u, v) = (u^\epsilon(\xi), v^\epsilon(\xi))$, are equivalent to the third-order equation

$$\rho U + (3u^\epsilon(\xi)^2 - x(\xi))U_\xi + 6u^\epsilon(\xi)u^\epsilon_\xi U = \alpha U_{\xi\xi} + U_{\xi\xi\xi}.$$

Compare (4.3).

For each $\rho \in \mathbb{C}$, the system (4.9)–(4.15) has the solution

$$(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi, 0, 0, 0). \tag{4.16}$$

Suppose for some ρ there is another solution

$$(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi, U(\xi), V(\xi), W(\xi))$$

such that $(U, V, W) \to (0, 0, 0)$ as $\xi \to \pm\infty$ and $U(\xi) \not\equiv 0$. Then in an appropriate function space, for $\lambda = \frac{\rho}{\epsilon}$, $\big(\lambda, U(\frac{x}{\epsilon})\big)$ is an eigenpair for the linear equation (4.5).

4.3. **Equilibria.** For $\epsilon = 0$ the set of equilibria of (4.9)–(4.15) includes ux-space. For $\rho \neq 0$ there are no other equilibria.

The derivative of the right hand side of (4.9)–(4.15), evaluated at an equilibrium in ux-space, has the matrix

$$J(u, 0, 0, x, 0, 0, 0, \rho) = \begin{pmatrix} A(u, x, 0) & 0 & 0 \\ 0 & 0 & A(u, x, \rho) \end{pmatrix},$$

where

$$A(u, x, \rho) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \rho & 3u^2 - x & -\alpha \end{pmatrix}. \tag{4.17}$$

The eigenvalues of $J(u, 0, 0, x, 0, 0, 0, \rho)$ are 0 and the eigenvalues of $A(u, x, 0)$ and $A(u, x, \rho)$. The eigenvalues of $A(u, x, 0)$ are 0 and $\mu^\pm(u, x)$ given by (2.9). The eigenvalues of $A(u, x, \rho)$ are the roots of $g(\mu, u, x, \rho) = \mu^3 + \alpha\mu^2 - (3u^2 - x)\mu - \rho$, which for fixed (u, x, ρ) is a cubic polynomial in μ.

$A(u, x, \rho)$ has an eigenvalue with 0 real part provided there is a real number b such that $g(bi, u, x, \rho) = 0$. This equation can be written

$$\rho = -\alpha b^2 - ib(b^2 + 3u^2 - x). \tag{4.18}$$

For fixed (u, x), the curve (4.18) lies $\{\rho : \operatorname{Re}\rho < 0\} \cup \{0\}$. Therefore for $\operatorname{Re}\rho \geq 0$ and $\rho \neq 0$, the number of eigenvalues of $A(u, x, \rho)$ with negative (respectively positive) real part never changes.

We shall determine these numbers by considering $\rho \in \mathbb{R}^+$, for which g is a real polynomial. Let the roots of g be μ_1, μ_2, μ_3. We have

$$-\alpha = \mu_1 + \mu_2 + \mu_3, \tag{4.19}$$

$$\rho = \mu_1 \mu_2 \mu_3. \tag{4.20}$$

From the fact that g is a real polynomial, $\rho > 0$, and (4.20), we see that at least one μ_i, say μ_3, must be positive. Then using (4.19) and (4.20), we conclude that μ_1 and μ_2 have negative real part. Hence for $\operatorname{Re}\rho \geq 0$ and $\rho \neq 0$, $A(u, x, \rho)$ has two eigenvalues with negative real part and one with positive real part.

4.4. **Invariant manifolds.** In the situation of Theorem 2.3 or 2.4, let us fix ρ with $\operatorname{Re}\rho \geq 0$ and $\rho \neq 0$.

In $uvwxUVW$-space, for a small $\delta > 0$, let

$$\mathcal{K} = \{(u, 0, 0, x, 0, 0, 0) : |u| \leq \frac{1}{\delta} \text{ and } -\infty < x \leq 3u^2 - \delta\},$$

$$\mathcal{K}^{u^*} = \{(u, 0, 0, x, 0, 0, 0) : u = u^* \text{ and } -\infty < x \leq 3(u^*)^2 - \delta\} \subset \mathcal{K},$$

$$\mathcal{L} = \{(u, 0, 0, x, 0, 0, 0) : |u| \leq \frac{1}{\delta} \text{ and } 3u^2 + \delta \leq x < \infty\},$$

$$\mathcal{L}^{u^*} = \{(u, 0, 0, x, 0, 0, 0) : u = u^* \text{ and } 3(u^*)^2 + \delta \leq x < \infty\} \subset \mathcal{L}.$$

For $\epsilon = 0$, $\mathcal{K}$ and $\mathcal{L}$ can be viewed as 2-dimensional normally hyperbolic manifolds of equilibria for the system (4.9)–(4.15). (See Appendix A for how to deal with the noncompactness.) Each point of $\mathcal{K}$ has a 2-dimensional unstable manifold and a 3-dimensional stable manifold; each point of $\mathcal{L}$ has a 1-dimensional unstable manifold and a 4-dimensional stable manifold. It follows easily that for the system (4.9)–(4.15), for small ϵ, $\mathcal{K}$ and $\mathcal{L}$ are 2-dimensional normally hyperbolic invariant manifolds. Each point of $\mathcal{K}$ has a 2-dimensional unstable fiber and a 3-dimensional stable fiber; each point of $\mathcal{L}$ has a 1-dimensional unstable fiber and a 4-dimensional stable fiber. $\mathcal{K}^{u^*}$ and $\mathcal{L}^{u^*}$ remain invariant for $\epsilon \neq 0$. For small ϵ, $\mathcal{K}^{u^*}$ has 3-dimensional unstable and 4-dimensional stable manifolds, denoted $W^u_\epsilon(\mathcal{K}^{u^*})$ and $W^s_\epsilon(\mathcal{K}^{u^*})$ respectively; and $\mathcal{L}^{u^*}$ has 2-dimensional unstable and 5-dimensional stable manifolds, denoted $W^u_\epsilon(\mathcal{L}^{u^*})$ and $W^s_\epsilon(\mathcal{L}^{u^*})$ respectively.

4.5. **Eigenvalues.** For small $\epsilon > 0$, the intersection of $W^u_\epsilon(\mathcal{K}^{u^\ell})$ and $W^s_\epsilon(\mathcal{L}^{u^r})$ includes the solution (4.16) of (4.9)–(4.15). If $W^u_\epsilon(\mathcal{K}^{u^\ell})$ and $W^s_\epsilon(\mathcal{K}^{u^r})$ meet transversally along this solution, then $\lambda = \frac{\rho}{\epsilon}$ is not an eigenvalue of (4.5) in any space of bounded solutions, or in L^1 or L^2.

Theorem 4.2. *Let $\alpha > 0$, and let (u^ℓ, u^r) satisfy the conditions of Theorem 2.3 (respectively, Theorem 2.4). Assume the viscous profile is spectrally stable (respectively, the two viscous profiles are spectrally stable). Fix ρ with $\operatorname{Re}\rho \geq 0$ and $\rho \neq 0$. Then there exists $\epsilon(\alpha, \rho) > 0$ such that for $0 < \epsilon < \epsilon(\alpha, \rho)$, $W^u_\epsilon(\mathcal{K}^{u^\ell})$ and $W^s_\epsilon(\mathcal{K}^{u^r})$ meet transversally along the solution (4.16) of (4.9)–(4.15). $\epsilon(\alpha, \rho)$ depends continuously on (α, ρ) (and on (u^ℓ, u^r)).*

We shall prove Theorem 4.2 for the case in which the conditions of Theorem 2.4 are satisfied; the other case is even easier.

Proof. In $uvwxUVW$-space, let

$$\Gamma_1 = \{(u^\ell, 0, 0, x, 0, 0, 0) : -\infty < x \leq x^*\},$$

$$\Gamma_2 = \{(u^*(\eta), v^*(\eta), w^*(\eta), x^*, 0, 0, 0) : v^* = u^*_\eta,\ w^* = u^*_{\eta\eta},\ -\infty < \eta < \infty\},$$
$$\Gamma_3 = \{(u^m, 0, 0, x, 0, 0, 0) : x^* \le x \le x^\diamond\},$$
$$\Gamma_4 = \{(u^\diamond(\eta), v^\diamond(\eta), w^\diamond(\eta), x^\diamond, 0, 0, 0) : v^\diamond = u^\diamond_\eta,\ w^\diamond = u^\diamond_{\eta\eta},\ -\infty < \eta < \infty\},$$
$$\Gamma_5 = \{(u^r, 0, 0, x, 0, 0, 0) : x^\diamond \le x < \infty\}.$$

(We have used ξ instead of η for the traveling wave variable.) We claim:

(1) $W^u_0(\mathcal{K}^{u^\ell})$ and $W^s_0(\mathcal{K})$ intersect transversally along Γ_2.
(2) $W^u_0(\mathcal{K}^{u^m})$ and $W^s_0(\mathcal{L}^{u^r})$ intersect transversally along Γ_4.

Once the first claim is proved, we note that by the exchange lemma, for small $\epsilon > 0$, $W^u_\epsilon(\mathcal{K}^{u^\ell})$, followed forwards in time, arrives near the point $(u^m, 0, 0, x^\diamond, 0, 0, 0)$ C^1-close to $W^u_\epsilon(\mathcal{K}^{u^m})$. Then the second claim implies that $W^u_\epsilon(\mathcal{K}^{u^\ell})$ and $W^s_\epsilon(\mathcal{L}^{u^r})$ intersect transversally near Γ_4. The result follows.

The two claims have similar proofs; we prove only the first. Let $(\bar u, \bar v, \bar w, \bar x, \bar U, \bar V, \bar W)$ be vector that is tangent to both $W^u_0(\mathcal{K}^{u^\ell})$ and $W^s_0(\mathcal{K})$ at $(u^*(0), v^*(0), w^*(0), x^*, 0, 0, 0)$. Then

(a) $(\bar u, \bar v, \bar w, \bar x)$ is tangent to both $W^u_0(\mathcal{P}^{u^\ell})$ and $W^s_0(\mathcal{P})$ at $(u^*(0), v^*(0), w^*(0), x^*)$.
(b) The solution of the linear differential equation (4.13)–(4.15) with $(u, v, x) = (u^*(\xi), v^*(\xi), x^*)$ and $(U, V, W)(0) = (\bar U, \bar V, \bar W)$ approaches $(0, 0, 0)$ as $\xi \to \pm\infty$.

We will show:

(i) $(\bar u, \bar v, \bar w, \bar x)$ is a multiple of $(u^*_\xi(0), v^*_\xi(0), w^*_\xi(0), 0)$.
(ii) $(\bar U, \bar V, \bar W) = (0, 0, 0)$.

Statements (i) and (ii) imply that the tangent spaces to $W^u_0(\mathcal{K}^{u^\ell})$ and $W^s_0(\mathcal{K})$ at $(u^*(0), v^*(0), w^*(0), x^*, 0, 0, 0)$ have 1-dimensional intersection. Since $W^u_0(\mathcal{K}^{u^\ell})$ has dimension 3, $W^s_0(\mathcal{K})$ has dimension 5, and $uvwxUVW$-space has dimension 7, this proves transversality.

To prove (i) and (ii), note that the proof of Theorem 2.4 shows that (a) holds if and only if $(\bar u, \bar v, \bar w, \bar x)$ is a multiple of $(u^*_\xi(0), v^*_\xi(0), w^*_\xi(0), 0)$. If (b) held for some $(\bar U, \bar V, \bar W) \neq (0, 0, 0)$, then, since the convergence would be exponential, equation (4.3), with $s = x^*$, would have a nontrivial solution in L^2, namely $U(\eta)$. This would contradict our assumption that $u^*(\xi)$ is spectrally stable. □

5. Eigenvalues of order one

Substituting $\rho = \epsilon\lambda$ back into the system (4.9)–(4.15) yields the system

$$u_\xi = v, \tag{5.1}$$
$$v_\xi = w, \tag{5.2}$$
$$w_\xi = (3u^2 - x)v - \alpha w, \tag{5.3}$$
$$x_\xi = \epsilon, \tag{5.4}$$
$$U_\xi = V, \tag{5.5}$$
$$V_\xi = W, \tag{5.6}$$
$$W_\xi = \epsilon\lambda U + (3u^2 - x)V + 6uvU - \alpha W. \tag{5.7}$$

For each $\lambda \in \mathbb{C}$, the system (5.1)–(5.7) has the solution

$$(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi, 0, 0, 0). \tag{5.8}$$

Suppose for some λ there is another solution

$$(u^\epsilon(\xi), v^\epsilon(\xi), w^\epsilon(\xi), \epsilon\xi, U(\xi), V(\xi), W(\xi))$$

such that $(U,V,W) \to (0,0,0)$ as $\xi \to \pm\infty$ and $U(\xi) \not\equiv 0$. Then in an appropriate function space, $\big(\lambda, U(\frac{x}{\epsilon})\big)$ is an eigenpair for the linear equation (4.5).

5.1. **Equilibria and invariant manifolds.** For $\epsilon = 0$, the set of equilibria of (5.1)–(5.7) is uxU-space. The derivative of the right-hand side of (5.1)–(5.7), evaluated at an equilibrium, has the matrix

$$\begin{pmatrix} A(u,x,0) & 0 & 0 \\ 0 & 0 & A(u,x,0) \end{pmatrix},$$

where $A(u,x,0)$ is given by (4.17) with $\rho = 0$. The eigenvalues are 0 with multiplicity 3 and $\mu^\pm(u,x)$ given by (2.9), each with multiplicity 2. So for $x < 3u^2$ there are two positive eigenvalues and two negative eigenvalues; for $x > 3u^2$, there are four eigenvalues with negative real part.

In the situation of Theorem 2.3 or 2.4, let us fix $\lambda \in \mathbb{C}$.

In $uvwxUVW$-space, for a small $\delta > 0$, let

$$\mathcal{M}_0 = \{(u,0,0,x,U,0,0) : |u| \le \frac{1}{\delta} \text{ and } -\infty < x \le 3u^2 - \delta\},$$

$$\mathcal{N}_0 = \{(u,0,0,x,U,0,0) : |u| \le \frac{1}{\delta} \text{ and } 3u^2 + \delta \le x < \infty\},$$

For $\epsilon = 0$, $\mathcal{M}_0$ and $\mathcal{N}_0$ can be viewed as 3-dimensional normally hyperbolic manifolds of equilibria for the system (5.1)–(5.7). (See Appendix A for how to deal with the noncompactness.) Each point of $\mathcal{M}_0$ has a 2-dimensional unstable manifold and a 2-dimensional stable manifold; each point of $\mathcal{N}_0$ has a 4-dimensional stable manifold. Therefore for the system (4.9)–(4.15), for small ϵ, $\mathcal{M}_0$ and $\mathcal{N}_0$ perturb to normally hyperbolic invariant manifolds $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ respectively. Each point of $\mathcal{M}_\epsilon$ has a 2-dimensional unstable fiber and a 2-dimensional stable fiber; each point of $\mathcal{N}_\epsilon$ has a 4-dimensional stable fiber.

In the remainder of this subsection we give some further information about the manifolds $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ that will be needed in Subsection 5.5.1.

The equations of $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ must take the form

$$v = 0, \tag{5.9}$$

$$w = 0, \tag{5.10}$$

$$V = a(u,x,\lambda,\epsilon)U = \big(\epsilon a^1(u,x,\lambda) + \mathcal{O}(\epsilon^2)\big)U, \tag{5.11}$$

$$W = b(u,x,\lambda,\epsilon)U = \big(\epsilon b^1(u,x,\lambda) + \mathcal{O}(\epsilon^2)\big)U. \tag{5.12}$$

From (5.11), on $\mathcal{M}_\epsilon$ or $\mathcal{N}_\epsilon$ we have

$$\begin{aligned} V_\xi &= \epsilon\left(\frac{\partial a^1}{\partial u}u_\xi + \frac{\partial a^1}{\partial x}x_\xi\right)U + \epsilon a^1\dot{U} + \mathcal{O}(\epsilon^2)U \\ &= \epsilon\left(\frac{\partial a^1}{\partial u}v + \frac{\partial a^1}{\partial x}\epsilon\right)U + \epsilon a^1 V + \mathcal{O}(\epsilon^2)U \\ &= \epsilon^2\frac{\partial a^1}{\partial x}U + (\epsilon a^1)^2 U + \mathcal{O}(\epsilon^2)U \\ &= \mathcal{O}(\epsilon^2)U. \end{aligned}$$

On the other hand, on $\mathcal{M}_\epsilon$ or $\mathcal{N}_\epsilon$, $V_\xi = W$ given by (5.12). We conclude that $b^1 = 0$. Then (5.12) implies that on $\mathcal{M}_\epsilon$ or $\mathcal{N}_\epsilon$, $W_\xi = \mathcal{O}(\epsilon^2)U$. On the other hand, from (5.7) we have

$$\begin{aligned} W_\xi &= \epsilon\lambda U + (3u^2 - x)V + 6uvU - \alpha W \\ &= \epsilon\lambda U + (3u^2 - x)\epsilon a^1 U - \alpha\epsilon b^1 U + \mathcal{O}(\epsilon^2)U \\ &= \epsilon\left(\lambda + (3u^2 - x)a^1\right)U + \mathcal{O}(\epsilon^2)U. \end{aligned}$$

We conclude that $a^1(u, x, \lambda) = \frac{\lambda}{x - 3u^2}$. Thus, the equations of $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ are (5.9)–(5.10) together with

$$V = \left(\epsilon\frac{\lambda}{x - 3u^2} + \mathcal{O}(\epsilon^2)\right)U, \tag{5.13}$$

$$W = \mathcal{O}(\epsilon^2)U. \tag{5.14}$$

The system (5.1)–(5.7), restricted to $\mathcal{M}_\epsilon$ or $\mathcal{N}_\epsilon$, reduces to

$$u_\xi = 0, \tag{5.15}$$

$$x_\xi = \epsilon, \tag{5.16}$$

$$U_\xi = \left(\epsilon\frac{\lambda}{x - 3u^2} + \mathcal{O}(\epsilon^2)\right)U. \tag{5.17}$$

5.2. **Statement of results.** Let

$$\mathcal{M}^{u^*} = \left\{(u, 0, 0, x, 0, 0, 0) : u = u^* \text{ and } -\infty < x \le 3(u^*)^2 - \delta\right\},$$
$$\mathcal{N}^{u^*} = \left\{(u, 0, 0, x, 0, 0, 0) : u = u^* \text{ and } 3(u^*)^2 + \delta \le x < \infty\right\}.$$

Notice that $\mathcal{M}^{u^*}$ and $\mathcal{N}^{u^*}$ have dimension 1 and are invariant subsets of $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ respectively for every ϵ. $\mathcal{M}^{u^*}$ has 3-dimensional unstable and 3-dimensional stable manifolds, denoted $W^u_\epsilon(\mathcal{M}^{u^*})$ and $W^s_\epsilon(\mathcal{M}^{u^*})$ respectively; $\mathcal{N}^{u^*}$ has a 5-dimensional stable manifold, denoted $W^s_\epsilon(\mathcal{N}^{u^*})$.

For small $\epsilon > 0$, the intersection of $W^u_\epsilon(\mathcal{M}^{u^\ell})$ and $W^s_\epsilon(\mathcal{N}^{u^r})$ includes the solution (5.8) of (5.1)–(5.7). If $W^u_\epsilon(\mathcal{M}^{u^\ell})$ and $W^s_\epsilon(\mathcal{N}^{u^r})$ meet transversally along this solution, then λ is not an eigenvalue of (4.5) in any space of bounded functions, or in L^1 or L^2.

Theorem 5.1. *Let $\alpha > 0$, and let (u^ℓ, u^r) satisfy the conditions of Theorem 2.3 (respectively, Theorem 2.4). Fix $\lambda \ne -1$ (respectively, $\lambda \ne -1$ and $\lambda \ne \lambda^*$, where λ^* is given by (5.39)). Then there exists $\epsilon(\lambda) > 0$ such that for $0 < \epsilon < \epsilon(\lambda)$, $W^u_\epsilon(\mathcal{M}^{u^\ell})$ and $W^s_\epsilon(\mathcal{N}^{u^r})$ meet transversally along the solution (5.8) of (5.1)–(5.7). $\epsilon(\lambda)$ depends continuously on λ (and on (α, u^ℓ, u^r)).*

Note that it is not necessary to assume that the individual viscous profiles are spectrally stable. The remainder of this section is devoted to the proofs of the two parts of this theorem. The proofs differ from those of the previous section in that there is no transversality at $\epsilon = 0$.

5.3. **Adjoint and inhomogeneous equations.** Let $\alpha > 0$, and let (u^ℓ, u^r) satisfy the conditions of Theorem 2.3 or Theorem 2.4). In the linear differential equation

$$U_\xi = V, \tag{5.18}$$

$$V_\xi = W, \tag{5.19}$$

$$W_\xi = (3u^*(\xi)^2 - x^*)V + 6u^*(\xi)v^*(\xi)U - \alpha W, \tag{5.20}$$

we make the change of variables

$$U = U, \quad V = V, \quad Z = (3u^*(\xi)^2 - x^*)U - \alpha V - W. \tag{5.21}$$

We obtain the linear system

$$X_\xi = \hat{A}(\xi)X, \quad X = \begin{pmatrix} U \\ V \\ Z \end{pmatrix}, \quad \hat{A}(\xi) = \begin{pmatrix} 0 & 1 & 0 \\ 3u^*(\xi)^2 - x^* & -\alpha & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.22}$$

The adjoint system is $\Psi_\xi = -\Psi \hat{A}(\xi)$, where Ψ is a row vector. It can be written

$$\begin{aligned} \Psi_{1\xi} &= (x^* - 3u^*(\xi)^2)\Psi_2, \\ \Psi_{2\xi} &= -\Psi_1 + \alpha\Psi_2, \\ \Psi_{3\xi} &= \Psi_2. \end{aligned}$$

One bounded solution is $\Psi^1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. Another solution is $\Psi^2 = \begin{pmatrix} \Psi_1^2 & \Psi_2^2 & \Psi_3^2 \end{pmatrix}$ with

$$\Psi_1^2(\xi) = -e^{\alpha\xi} v_\xi^*, \quad \Psi_2^2(\xi) = e^{\alpha\xi} u_\xi^*, \quad \Psi_3^2(\xi) = -\int_\xi^\infty \Psi_2^2(\zeta)\, d\zeta. \tag{5.23}$$

It is bounded when $(u^*(\xi), v^*(\xi))$ is a saddle-to-saddle solution (first wave of a two-wave Riemann solution) and unbounded when $(u^*(\xi), v^*(\xi))$ is a saddle-to-node solution (one-wave Riemann solution or second wave of a two-wave Riemann solution; in the latter case the asterisks should be replaced by diamonds).

Consider an inhomogeneous equation

$$X_\xi = \hat{A}(\xi)X + H, \tag{5.24}$$

Let $X(\xi)$ be a solution of (5.24), and let $\Psi(\xi)$ be a solution of $\Psi_\xi = -\Psi\hat{A}(\xi)$. Then

$$\Psi H = \Psi(X_\xi - \hat{A}(\xi)X) = \Psi X_\xi + \Psi_\xi X = (\Psi X)_\xi. \tag{5.25}$$

5.4. **Proof of Theorem 5.1 for one-wave Riemann solutions.** A basis for the tangent space to $W_0^u(\mathcal{M}^{u^\ell})$ at the point $(u^*(0), v^*(0), w^*(0), x^*, 0, 0, 0)$ consists of the vector $X^1 = (u_\xi^*(0), v_\xi^*(0), w_\xi^*(0), 0, 0, 0, 0)$, a second vector of the form $X^2 = (*, *, *, 1, 0, 0, 0)$, and $Y^1 = (0, 0, 0, 0, u_\xi^*(0), v_\xi^*(0), w_\xi^*(0))$. A basis for the tangent space to $W_0^s(\mathcal{N}^{u^r})$ at the same point consists of the vector $X^3 = X^1$, a linearly independent vector of the form $X^4 = (*, *, *, 0, 0, 0, 0)$, a vector of the form $X^5 = (*, *, *, 1, 0, 0, 0)$, the vector $Y^2 = Y^1$, and a linearly independent vector of the form $Y^3 = (0, 0, 0, 0, *, *, *)$. (For $X^1, \ldots, X^5$, see (2.32).) These eight vectors span a 6-dimensional subspace of $\mathbb{R}^7$. For a given λ, they perturb to vectors $X^i(\epsilon)$ and $Y^i(\epsilon)$ tangent to $W_\epsilon^u(\mathcal{M}^{u^\ell})$ or $W_\epsilon^s(\mathcal{N}^{u^r})$ at $(u(0,\epsilon), v(0,\epsilon), w(0,\epsilon), x(0,\epsilon), 0, 0, 0)$, with $X^i(\epsilon)$ in $\mathbb{R}^4 \times \{0\}$ and $Y^i(\epsilon)$ in $\{0\} \times \mathbb{R}^3$. These vectors span $\mathbb{R}^7$ if and only if the vectors $Y^i(\epsilon)$ span $\{0\} \times \mathbb{R}^3$.

Let $D(\epsilon) = \det(Y^1(\epsilon), Y^2(\epsilon), Y^3(\epsilon))$. Then $D(0) = 0$ and

$$\begin{aligned} D'(0) = \det((Y^1)'(0), Y^2(0), Y^3(0)) + \det(Y^1(0), (Y^2)'(0), Y^3(0)) \\ + \det(Y^1(0), Y^2(0), (Y^3)'(0)) = \det((Y^1)'(0) - (Y^2)'(0), Y^2(0), Y^3(0)). \end{aligned}$$

We shall show that for $\lambda \neq -1$, $D'(0) \neq 0$. This fact implies Theorem 5.1 in the case of a one-wave Riemann solution.

Let $Y^i(\epsilon) = (U^i(0,\epsilon), V^i(0,\epsilon), W^i(0,\epsilon))$, $i = 1, 2, 3$,

$$U^i(\xi, \epsilon) = U_0^i(\xi) + \epsilon U_1^i(\xi) + \mathcal{O}(\epsilon^2), \tag{5.26}$$

$$V^i(\xi,\epsilon) = V_0^i(\xi)+\epsilon V_1^i(\xi)+\mathcal{O}(\epsilon^2), \tag{5.27}$$
$$W^i(\xi,\epsilon) = W_0^i(\xi)+\epsilon W_1^i(\xi)+\mathcal{O}(\epsilon^2). \tag{5.28}$$

Then

$$D'(0)=\det\begin{pmatrix} U_1^1(0)-U_1^2(0) & U_0^2(0) & U_0^3(0)\\ V_1^1(0)-V_1^2(0) & V_0^2(0) & V_0^3(0)\\ W_1^1(0)-W_1^2(0) & W_0^2(0) & W_0^3(0)\end{pmatrix} \tag{5.29}$$

We apply the linear change of variables (5.21) with $\xi=0$ to the columns of this matrix and obtain the same matrix with the third row replaced by $\big(Z_1^1(0)-Z_1^2(0)\quad Z_0^2(0)\quad Z_0^3(0)\big)$. The determinant (5.29) is nonzero if and only if the determinant of the new matrix is nonzero. We shall see that $Z_0^2(0)=Z_0^3(0)=0$ and the top right 2×2 block is nonsingular. Hence $D'(0)\neq 0$ provided $Z_1^1(0)-Z_1^2(0)\neq 0$.

For each i, $(U_0^i(\xi),V_0^i(\xi),W_0^i(\xi))$ is a solution of (5.18)–(5.20). Applying the change of variables (5.21), we obtain solutions $(U_0^i(\xi),V_0^i(\xi),Z_0^i(\xi))$ of (5.22).

Note that from their definition, $(U_0^i(\infty),V_0^i(\infty),W_0^i(\infty))=(0,0,0)$ for all i. Then from (5.21), $Z_0^i(\infty)=0$ for all i. It follows from (5.25) with $\Psi=\Psi^1$, $H=0$, and $X=(U_0^i,V_0^i,Z_0^i)$ that $Z_0^i(\xi)\equiv 0$ for all i. Hence $Z_0^1(0)=Z_0^2(0)=Z_0^3(0)=0$.

We now consider (5.26)–(5.28) with $i=1$ or 2, so $(U_0^i(\xi),V_0^i(\xi),W_0^i(\xi))=(u_\xi^*,v_\xi^*,w_\xi^*)$. We drop the superscript for simplicity. Substituting into (5.5)–(5.7) and simplifying, we obtain at order ϵ:

$$U_{1\xi} = V_1, \tag{5.30}$$
$$V_{1\xi} = W_1, \tag{5.31}$$
$$\begin{aligned} W_{1\xi} &= (3u^*(\xi)^2-x^*)V_1+6u^*(\xi)v^*(\xi)U_1-\alpha W_1\\ &\quad -x_1(\xi)v_\xi^*+\big(\lambda u^*(\xi)+6u^*(\xi)v^*(\xi)u_1(\xi)\big)_\xi. \end{aligned} \tag{5.32}$$

This system is satisfied by $(U_1^i(\xi),V_1^i(\xi),W_1^i(\xi))$ for $i=1,2$. As $\xi\to-\infty$ (respectively, $\xi\to\infty$), $(u(\xi,\epsilon),v(\xi,\epsilon),w(\xi,\epsilon),x(\xi,\epsilon),U^1(\xi,\epsilon),V^1(\xi,\epsilon),W^1(\xi,\epsilon))$ (respectively, $(u(\xi,\epsilon),$ $v(\xi,\epsilon),w(\xi,\epsilon),x(\xi,\epsilon),U^2(\xi,\epsilon),V^2(\xi,\epsilon),W^2(\xi,\epsilon))$) converges to a solution in $\mathcal{M}^{u^\ell}$ (respectively, $\mathcal{N}^{u^r}$), for which we must have $(U,V,W)=(0,0,0)$. Therefore

$$\lim_{\xi\to-\infty}(U_1^1(\xi),V_1^1(\xi),W_1^1(\xi))=(0,0,0) \text{ and } \lim_{\xi\to\infty}(U_1^2(\xi),V_1^2(\xi),W_1^2(\xi))=(0,0,0). \tag{5.33}$$

In (5.30)–(5.32) we again make the change of variables (5.21). We obtain the nonhomogenous linear system

$$U_{1\xi} = V_1, \tag{5.34}$$
$$V_{1\xi} = (3(u^*(\xi))^2-x^*)U_1-\alpha V_1-Z_1, \tag{5.35}$$
$$Z_{1\xi} = x_1(\xi)v_\xi^*-\big(\lambda u^*(\xi)+6u^*(\xi)v^*(\xi)u_1(\xi)\big)_\xi. \tag{5.36}$$

For $i=1,2$, this system is satisfied by $(U_1^i(\xi),V_1^i(\xi),Z_1^i(\xi))$ corresponding to $(U_1^i(\xi),V_1^i(\xi),$ $W_1^i(\xi))$ under the change of variables. Using (5.33), we have $Z_1^1(-\infty)=Z_1^2(\infty)=0$. Using this fact and the fact that $u_1(\pm\infty)=0$ (see Subsection 3.1), we have

$$\begin{aligned} Z_1^1(0)-Z_1^2(0) &= \int_{-\infty}^{\infty} x_1(\xi)v_\xi^*-\lambda u_\xi^*\,d\xi = -\int_{-\infty}^{\infty} v^*(\xi)\,d\xi-\lambda(u^r-u^\ell)\\ &= -(u^r-u^\ell)-\lambda(u^r-u^\ell)=-(\lambda+1)(u^r-u^\ell). \end{aligned} \tag{5.37}$$

Since $u^r \neq u^\ell$, $Z_1^1(0) - Z_1^2(0) \neq 0$ for $\lambda \neq -1$. This proves Theorem 5.1 in the case of one-wave Riemann solutions.

5.5. **Proof of Theorem 5.1 for two-wave Riemann solutions.**

5.5.1. *First wave.* A basis for the tangent space to $W_0^u(\mathcal{M}^{u^\ell})$ at the point $(u^*(0), v^*(0), w^*(0), x^*, 0, 0, 0)$ was given in the previous subsection. A basis for the tangent space to $W_0^s(\mathcal{M}_0)$ at the same point consists of the vector $X^3 = X^1$, a linearly independent vector of the form $X^4 = (*, *, *, 0, 0, 0, 0)$, a vector of the form $X^5 = (*, *, *, 1, 0, 0, 0)$, the vector $Y^2 = Y^1$, and a linearly independent vector of the form $Y^3 = (0, 0, 0, 0, *, *, *)$. These eight vectors span a 6-dimensional subspace of $\mathbb{R}^7$. For a given λ, they perturb to vectors $X^i(\epsilon)$ and $Y^i(\epsilon)$ tangent to $W_\epsilon^u(\mathcal{M}^{u^\ell})$ or $W_\epsilon^s(\mathcal{M})$ at $(u(0,\epsilon), v(0,\epsilon), w(0,\epsilon), x(0,\epsilon), 0, 0, 0)$, with $X^i(\epsilon)$ in $\mathbb{R}^4 \times \{0\}$ and $Y^i(\epsilon)$ in $\{0\} \times \mathbb{R}^3$. As in the previous subsection, these vectors span $\mathbb{R}^7$ for small $\epsilon > 0$ provided

$$\det((Y^1)'(0) - (Y^2)'(0), Y^2(0), Y^3(0)) \neq 0. \tag{5.38}$$

After deriving (5.30)–(5.32) as in the previous subsection and making the change of variables (5.21), we write the system (5.34)–(5.36) as (5.24) with

$$H(\xi) = (0, 0, x_1(\xi)v_\xi^* - \big(\lambda u^*(\xi) + 6u^*(\xi)v^*(\xi)u_1(\xi)\big)_\xi).$$

Now $\Psi^2(\xi)Y^2(\xi)$ and $\Psi^2(\xi)Y^3(\xi)$ are identically 0; this follows from the fact that that $\Psi^i(\xi)Y^j(\xi)$ is constant by (5.25) with $H = 0$, together with $\Psi^2(\infty) = 0$, $Y^2(\infty) = 0$, and $Y^3(\infty)$ is a constant vector. (However, $\Psi^3(\infty)Y^3(\infty)$ is not 0.) From (5.25) it follows that (5.38) holds if and only if the following is nonzero:

$$\Psi^2(0)\left((U_1^1, V_1^1, Z_1^1)(0) - (U_1^2, V_1^2, Z_1^2)(0)\right) =$$
$$\Psi^2(-\infty)(U_1^1, V_1^1, Z_1^1)(-\infty) + \int_{-\infty}^0 \Psi^2 H\, d\xi - \left(\Psi^2(\infty)(U_1^2, V_1^2, Z_1^2)(\infty) + \int_\infty^0 \Psi^2 H\, d\xi\right).$$

Now $\Psi^2(-\infty) = (0, 0, K)$ and $(U_1^1, V_1^1, Z_1^1)(-\infty) = (0, 0, 0)$, so $\Psi^2(-\infty)(U_1^1, V_1^1, Z_1^1)(-\infty) = 0$. To calculate $\Psi^2(\infty)(U_1^2, V_1^2, Z_1^2)(\infty)$, we note that in UVW-coordinates, $u(\xi,\epsilon), v(\xi,\epsilon), w(\xi,\epsilon), x(\xi,\epsilon), U^2(\xi,\epsilon), V^2(\xi,\epsilon), W^2(\xi,\epsilon))$ must approach a solution in $\mathcal{M}^\epsilon$, which we denote $(u^c(\xi,\epsilon), v^c(\xi,\epsilon), w^c(\xi,\epsilon), x^c(\xi,\epsilon), U^c(\xi,\epsilon), V^c(\xi,\epsilon), W^c(\xi,\epsilon))$. Write

$$(U^c(\xi,\epsilon), V^c(\xi,\epsilon), W^c(\xi,\epsilon) = (U_0^c(\xi), V_0^c(\xi), W_0^c(\xi)) + \epsilon(U_1^c(\xi), V_1^c(\xi), W_1^c(\xi)) + \dots.$$

Since $U^2(\xi,\epsilon) = u_\xi^*(\xi) + \epsilon U_1^2(\xi) + \dots$, we have $U_0^c(\xi) = 0$. This fact and (5.13) imply $V_0^c(\xi) = V_1^c(\xi) = 0$, and (5.14) implies $W_0^c(\xi) = W_1^c(\xi) = 0$. Moreover, $U_0^c(\xi) = 0$ and (5.17) imply that $U_1^c(\xi)$ is a constant, say $U_1^c(\xi) = K_1^c$. Therefore $(U_1^2, V_1^2)(\infty) = (K_1^c, 0)$, and from (5.21),

$$Z_1^2(\infty) = (3u^*(\infty)^2 - x^*)U_1^2(\infty) - \alpha V_1^2(\infty) - W_1^2(\infty) = (3u^*(\infty)^2 - x^*)K_1^c.$$

Since $\Psi^2(\infty) = (0, 0, 0)$, we see that $\Psi^2(\infty)(U_1^2, V_1^2, Z_1^2)(\infty) = 0$.

We conclude

$$\Psi^2(0)\left((U_1^1, V_1^1, Z_1^1)(0) - (U_1^2, V_1^2, Z_1^2)(0)\right) = \int_{-\infty}^\infty \Psi^2 H\, d\xi$$
$$= \int_{-\infty}^\infty \Psi_3^2(\xi)\left(x_1(\xi)v_\xi^* - \big(\lambda u^*(\xi) + 6u^*(\xi)v^*(\xi)u_1(\xi)\big)_\xi\right) d\xi$$

$$= -\lambda \int_{-\infty}^{\infty} \Psi_3^2(\xi) v^*(\xi)\, d\xi + \int_{-\infty}^{\infty} \Psi_3^2(\xi) \left(x_1(\xi) v_\xi^* - \left(6u^*(\xi)v^*(\xi)u_1(\xi)\right)_\xi \right) d\xi$$

$$= -\lambda \int_{-\infty}^{\infty} \Psi_3^2(\xi) v^*(\xi)\, d\xi + \int_{-\infty}^{\infty} \Psi_3^2(\xi) x_1(\xi) v_\xi^* + 6\Psi_2^2(\xi) u^*(\xi) v^*(\xi) u_1(\xi)\, d\xi$$

Note that

$$\int_{-\infty}^{\infty} \Psi_3^2(\xi) x_1(\xi) v_\xi^* d\xi = \int_{-\infty}^{\infty} \Psi_3^2(\xi) \left(\frac{d}{d\xi}(x_1(\xi)v^*(\xi)) - v^*(\xi) \right) d\xi$$

$$= -\int_{-\infty}^{\infty} \Psi_2^2(\xi) x_1(\xi) v^*(\xi)\, d\xi - \int_{-\infty}^{\infty} \Psi_3^2(\xi) v^*(\xi) d\xi.$$

Therefore $\Psi^2(0)\left((U_1^1, V_1^1, Z_1^1)(0) - (U_1^2, V_1^2, Z_1^2)(0)\right) \neq 0$ provided

$$\lambda \neq \lambda^* = \frac{\int_{-\infty}^{\infty} 6\Psi_2^2(\xi) u^*(\xi) v^*(\xi) u_1(\xi) + \Psi_3^2(\xi) x_1(\xi) w^*(\xi)\, d\xi}{\int_{-\infty}^{\infty} \Psi_3^2(\xi) v^*(\xi)\, d\xi}$$

$$= \frac{\int_{-\infty}^{\infty} \Psi_2^2(\xi) \left(6u^*(\xi)u_1(\xi) - x_1(\xi)\right) v^*(\xi) d\xi}{\int_{-\infty}^{\infty} \Psi_3^2(\xi) v^*(\xi)\, d\xi} - 1. \quad (5.39)$$

Theorem 2.1 (2) implies that $v^*(\xi) < 0$. Also

$$\Psi_3^2(\xi) = -\int_{\xi}^{\infty} \Psi_2^2(\tau) d\tau = -\int_{\xi}^{\infty} e^{\alpha\tau} v^*(\tau) d\tau > 0.$$

Therefore the denominator of the fraction in (5.39) is negative.

The computation of $u_1(\xi)$, which is needed in the formula for λ^*, was discussed in 3.2. We recall from Proposition 3.2 that $u_1(\xi)$ is only unique up to addition of a multiple of v^*. Thus in order for λ^* to be well-defined by (5.39), we must have

$$\int_{-\infty}^{\infty} \Psi_2^2(\xi) u^*(\xi) (v^*(\xi))^2 d\xi = \int_{-\infty}^{\infty} e^{\alpha\xi} u^*(\xi) (v^*(\xi))^3 d\xi = 0.$$

As a check on our work, we have independently verified this formula using Maple. The verification uses the fact that $u^*(\xi)$ can be found explicitly from the fact that the connecting orbit lies on an invariant parabola; it is

$$u^*(\xi) = \frac{u^m + u^\ell e^{-\frac{\sqrt{2}}{2}(u^\ell - u^m)\xi}}{1 + e^{-\frac{\sqrt{2}}{2}(u^\ell - u^m)\xi}}. \quad (5.40)$$

5.5.2. *Second wave.* A basis for the tangent space to $W_0^u(\mathcal{M})$ at the point $(u^\diamond(0), v^\diamond(0), w^\diamond(0), x^\diamond, 0, 0, 0)$ consists of the vector $X^1 = (u_\xi^\diamond(0), v_\xi^\diamond(0), w_\xi^\diamond(0), 0, 0, 0, 0)$, a linearly independent vector of the form $X^2 = (*, *, *, 0, 0, 0, 0)$, a vector of the form $X^3 = (*, *, *, 1, 0, 0, 0)$, the vector $Y^1 = (0, 0, 0, 0, u_\xi^\diamond(0), v_\xi^\diamond(0), w_\xi^\diamond(0))$, and a linearly independent vector of the form $Y^2 = (0, 0, 0, 0, *, *, *)$. A basis for the tangent space to $W_0^s(\mathcal{N}^{u^r})$ at the same point consists of the vector $X^4 = X^1$, a linearly independent vector of the form $X^5 = (*, *, *, 0, 0, 0, 0)$, a vector of the form $X^6 = (*, *, *, 1, 0, 0, 0)$, the vector $Y^3 = Y^1$, and a linearly independent vector of the form $Y^4 = (0, 0, 0, 0, *, *, *)$. These ten vectors span $\mathbb{R}^7$, so $W_0^u(\mathcal{M})$ and $W_0^s(\mathcal{N}^{u^r})$ meet transversally along $(u^\diamond(\xi), v^\diamond(\xi), w^\diamond(\xi), x^\diamond, 0, 0, 0)$. However, the usual exchange lemma cannot be used to follow $W_0^s(\mathcal{N}^{u^r})$ backwards, because the $W_0^s(\mathcal{N}^{u^r}) \cap W_0^u(u^m, 0, 0, x^\diamond, 0, 0, 0)$ is 2-dimensional.

However, let $(0,0,0,0,U(\xi,\epsilon),V(\xi,\epsilon),W(\xi,\epsilon))$ be a solution in $W^u_0(\mathcal{M})\cap W^s_\epsilon(\mathcal{N}^{u^r})$ given by (5.26)–(5.28). If $U_1(-\infty)\neq 0$, then the exchange lemma of [20], Section 4, if can be used for this purpose.

We have

$$Z_1(\infty)-Z_1(-\infty)=(3(u^r)^2-x^\diamond)U_1(\infty)-(3(u^m)^2-x^\diamond)U_1(-\infty)=-(3(u^m)^2-x^\diamond)U_1(-\infty), \tag{5.41}$$

and, using (5.32) and mimicking the calculation (5.37),

$$Z_1(\infty)-Z_1(-\infty)=\int_{-\infty}^{\infty} x_1(\xi)v^*_\xi-\lambda u^*_\xi\,d\xi=-(\lambda+1)(u^r-u^m). \tag{5.42}$$

From (5.41) and (5.42),

$$U_1(-\infty)=\frac{(\lambda+1)(u^r-u^m)}{3(u^m)^2-x^\diamond},$$

which is nonzero unless $\lambda=-1$.

5.5.3. *Completion of proof.* Let $x^m=\frac{1}{2}(x^*+x^\diamond)$. The idea is that for small $\epsilon>0$, (1) $W^u_\epsilon(\mathcal{M}^{u^\ell})$, followed forward, arrives near the point $(u,v,w,x,U,V,W)=(u^m,0,0,x^m,0,0,0)$ C^1-close to $W^u_\epsilon(\mathcal{M}^{u^m})$, and (2) $W^s_\epsilon(\mathcal{M}^{u^r})$, followed backward, arrives near the same point C^1-close to $W^s_\epsilon(\mathcal{M})$. Since $W^u_0(\mathcal{M}^{u^m})$ and $W^s_0(\mathcal{M})$ are transverse, so are $W^u_\epsilon(\mathcal{M}^{u^\ell})$ and $W^s_\epsilon(\mathcal{M}^{u^r})$.

The usual exchange lemma ([11], Theorem 6.5) cannot be used to show (1) because two of its hypotheses do not hold at $\epsilon=0$: (a) $W^u_0(\mathcal{M}^{u^\ell})$ and $W^s_0(\mathcal{M})$ are not transverse, as pointed out at the start of 5.5.1; and (b) at a point in the intersection of $W^u_0(\mathcal{M}^{u^\ell})$ and $W^s_0(u^m,0,0,x^*,0,0,0)$), the tangent spaces to these manifolds have 2-dimensional, rather than 1-dimensional, intersection (spanned by X^1 and Y^1 of 5.5.1). The calculation in 5.5.1 shows that both these failings are remedied for small $\epsilon>0$ at order ϵ.

The exchange lemma as stated in [8] only requires that the transversality and 1-dimensional intersection assumptions hold for small $\epsilon>0$; however, the result is not stated very precisely (it is not made clear at what order in ϵ the assumptions must hold), and for a proof the authors refer to the Brown thesis of S.-K. Tin, which has not been published. The paper that grew out of Tin's thesis, [11], states an exchange lemma (Theorem 8.3) in which the transversality assumption is only required to hold for small $\epsilon>0$ at algebraic order in ϵ, which is carefully defined. The 1-dimensional intersection assumption is only required to hold for small $\epsilon>0$, but the order is not specified, and the proof does not really refer to this assumption. Despite these lacunae in the literature, we shall simply assume that a version of the exchange lemma exists which can be used to prove (1).

The usual exchange lemma cannot be used to show (2) because one of its hypotheses does not hold at $\epsilon=0$: at a point in the intersection of $W^u_0(u^m,0,0,x^\diamond,0,0,0)$ and $W^s_0(\mathcal{M}^{u^r})$, the tangent spaces to these manifolds have 2-dimensional, rather than 1-dimensional, intersection (spanned by X^1 and Y^1 of 5.5.2). The calculation in 5.5.2 shows that both this failings is remedied for small $\epsilon>0$ at order ϵ. Because of the linear skew-product form of the system (5.1)–(5.7), the exchange lemma of [20], Theorem 4.1, implies (2).

5.6. **Numerical results.** For given (α,u^ℓ) with $u^\ell>\frac{2}{3}\alpha\sqrt{2}$, Theorem 2.1 allows one to choose x^* and u^m so that there is a saddle-to-saddle connection $(u^*(\xi),u^*_\xi(\xi))$ from $(u^\ell,0)$ to $(u^m,0)$, with $u^*(\xi)$ given by (5.40). The proof of Proposition 3.2 then shows how to compute $u^1(\xi)$ and x^1. Most of the computation can be done analytically, for arbitrary (α, u^ℓ), using Maple. In particular, a solution $X^2(\xi)$ of $X_\xi = A(\xi)X(\xi)$ that is independent of $(u^*_\xi, u^*_{\xi\xi})$ can be found analytically by reduction of order. However, toward the end of the computation of $u^1(\xi)$, one must assign values to (α, u^ℓ), use Maple's numerical routines, and replace infinite integrals by finite integrals. One can then compute λ^* from (5.39), again using Maple's numerical routines and replacing infinite integrals by finite integrals. We did this for $(\alpha, u^\ell) = (\frac{3\sqrt{2}}{2}, 3)$ and obtained $\lambda^* = -1.0000005$. This leads to the conjecture that in fact $\lambda^* = -1$.

6. Resolvent set

Consider the linear differential equation (4.4), which we now denote

$$U_t = \mathcal{E}^\epsilon U = \epsilon^2 U_{xxx} + \alpha\epsilon U_{xx} - (3(\hat{u}^\epsilon(x))^2 - x)U_x - 6\hat{u}^\epsilon(x)\hat{u}^\epsilon_x U. \tag{6.1}$$

In this section we are interested in resolvent values λ of the operator $\mathcal{E}_\epsilon$.

We define

$$\mathcal{S}^\epsilon U = \epsilon\mathcal{E}^\epsilon U = \epsilon^3 U_{xxx} + \alpha\epsilon^2 U_{xx} - \epsilon(3(\hat{u}^\epsilon(x))^2 - x)U_x - 6\epsilon\hat{u}^\epsilon(x)\hat{u}^\epsilon_x U. \tag{6.2}$$

The substitution $x = \epsilon\xi$ in (6.2) yields

$$\mathcal{T}^\epsilon U = U_{\xi\xi\xi} + \alpha U_{\xi\xi} - (3(u^\epsilon(\xi))^2 - \epsilon\xi)U_\xi - 6u^\epsilon(\xi)u^\epsilon_\xi U. \tag{6.3}$$

We shall view $\mathcal{T}^\epsilon$ as a linear operator on the weighted space $C(\epsilon\gamma, \mathbb{R}_\xi)$, the space of continuous functions from $\mathbb{R}$ (with variable ξ) to $\mathbb{R}$ such that the weighted norm $||U||_{\epsilon\gamma} = \sup_\xi |U(\xi)|e^{\epsilon\gamma|\xi|}$ is finite.

If $U(\xi) \in C(\epsilon\gamma, \mathbb{R}_\xi)$, then the function $\tilde{U}(x) = U\left(\frac{x}{\epsilon}\right)$ is in $C(\gamma, \mathbb{R}_x)$, the space of continuous functions from $\mathbb{R}$ (with variable x) to $\mathbb{R}$ such that the weighted norm $||\tilde{U}||_\gamma = \sup_x |\tilde{U}(x)|e^{\gamma|x|}$ is finite.

Note that ρ is an eigenvalue (respectively resolvent value) of $\mathcal{T}^\epsilon$ on $C(\epsilon\gamma, \mathbb{R}_\xi)$ if and only if ρ is an eigenvalue (respectively resolvent value) of $\mathcal{S}^\epsilon$ on $C(\gamma, \mathbb{R}_x)$, which is true if and only if $\lambda = \frac{\rho}{\epsilon}$ is an eigenvalue (respectively resolvent value) of $\mathcal{E}^\epsilon$ on $C(\gamma, \mathbb{R}_x)$.

The complex number ρ is in the resolvent set of $\mathcal{T}^\epsilon$ on $C(\epsilon\gamma, \mathbb{R}_\xi)$ if for each f in $C(\epsilon\gamma, \mathbb{R}_\xi)$, the nonhomogeneous problem

$$(\mathcal{T}^\epsilon - \rho I)U = f \tag{6.4}$$

has a unique solution U in $C(\epsilon\gamma, \mathbb{R}_\xi)$, and the mapping $U = (\mathcal{T}^\epsilon - \rho I)^{-1}f$ from $C(\epsilon\gamma, \mathbb{R}_\xi)$ to itself is bounded.

Let $\mathbb{C}_\delta = \{\rho : \operatorname{Re}\rho \geq -\delta\}$.

Theorem 6.1. *Let $\alpha > 0$, let (u^ℓ, u^r) satisfy the conditions of Theorem 2.3 or 2.4, and let Ω be a compact subset of $\mathbb{C}$. Let $0 < \delta_1 < \frac{\alpha^3}{192}$. Let m, ϵ_1, and L be positive constants given by Theorem 6.8 below. Let $\gamma > L + \frac{48}{L\alpha^2}$. Let $\delta_0 = \min\left(\frac{\epsilon_1}{L}, \delta_1\right)$ and let $\epsilon_0 = L\delta_0$. If $0 < \epsilon \leq \epsilon_0$, $\delta = \frac{\epsilon}{L}$, and $\rho \in \mathbb{C}_\delta \cap \Omega$, then ρ is an eigenvalue of $\mathcal{T}^\epsilon$ on $C(\epsilon\gamma, \mathbb{R}_\xi)$ with geometric multiplicity 1, or ρ is in the resolvent set of $\mathcal{T}^\epsilon$.*

Corollary 6.2. *Assume the hypotheses of Theorem 6.1, and define δ_0 and ϵ_0 as in that theorem. If $0 < \epsilon \leq \epsilon_0$, $\delta = \frac{\epsilon}{L}$, and $\lambda \in \mathbb{C}_{\frac{1}{L}} \cap \frac{1}{\epsilon}\Omega$, then λ is an eigenvalue of $\mathcal{E}^\epsilon$ on $C(\gamma, \mathbb{R}_x)$ with geometric multiplicity 1, or λ is in the resolvent set of $\mathcal{E}^\epsilon$.*

In this corollary, if we replace the set $\frac{1}{\epsilon}\Omega$ by a fixed compact subset of $\mathbb{C}$, then from Section 5, any eigenvalues must be near -1 or λ^*.

To prove Theorem 6.1, define the matrix

$$B(u,v,x,\rho) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \rho + 6uv & 3u^2 - x & -\alpha \end{pmatrix},$$

so that $A(u,x,\rho)$ defined by (4.17) is just $B(u,0,x,\rho)$. Let

$$\hat{B}(\rho,\epsilon,\xi) = B(u^\epsilon(\xi), v^\epsilon(\xi), \epsilon\xi, \rho) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \rho + 6u^\epsilon(\xi)v^\epsilon(\xi) & 3(u^\epsilon(\xi))^2 - \epsilon\xi & -\alpha \end{pmatrix}.$$

The system of linear equations (4.13)–(4.15), with $(u,v,x) = (u^\epsilon(\xi), v^\epsilon(\xi), \epsilon\xi)$, is just

$$Y_\xi = \hat{B}(\rho,\epsilon,\xi)Y. \tag{6.5}$$

The equation (6.4) can be written as

$$Y_\xi = \hat{B}(\rho,\epsilon,\xi)Y + F, \quad Y = (U,V,W), \quad F = (0,0,f). \tag{6.6}$$

For a fixed u^* we let

$$R(u^*,u,v) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6uv & 3\left(u^2 - (u^*)^2\right) & 0 \end{pmatrix}$$

Then for any u^*, $\hat{B}(\rho,\epsilon,\xi) = A(u^*,\epsilon\xi,\rho) + R(u^*, u^\epsilon(\xi), v^\epsilon(\xi))$.

Instead of immediately studying the linear system $Y_\xi = \hat{B}(\rho,\epsilon,\xi)Y$, we will study the simpler linear system $Y_\xi = A(u^*,\epsilon\xi,\rho)$, with u^* fixed. For $m > 0$, let

$$J_1(u,m) = (-\infty, 3u^2 - m], \quad J_2(u,m) = [3u^2 + m, \infty).$$

Let $J_j^\epsilon(u,m) = \{\xi : \epsilon\xi \in J_j(u,m)\}$. We will show (see Lemma 6.9) that a diagonalized version of $Y_\xi = A(u^*,\epsilon\xi,\rho)$ has pseudoexponential dichotomies (defined in Subsection 6.4) on $J_1^\epsilon(u^*,m)$ for m sufficiently large, and on $J_2^\epsilon(u^*,\frac{\alpha^2}{3})$. We will then use this fact and Coppel's Roughness Theorem (see Subsection 6.4) to show that in the coordinates in which $Y_\xi = A(u^\ell,\epsilon\xi,\rho)$ (respectively $Y_\xi = A(u^r,\epsilon\xi,\rho)$) is almost diagonalized, the system $Y_\xi = \hat{B}(\rho,\epsilon,\xi)Y$ has a pseudoexponential dichotomy on $J_1^\epsilon(u^*,m)$ (respectively $J_2^\epsilon(u^*,\frac{\alpha^2}{3})$). Part of the reason is that $R(u^\ell, u^\epsilon(\xi), v^\epsilon(\xi))$ (respectively $R(u^r, u^\epsilon(\xi), v^\epsilon(\xi))$) approaches 0 exponentially as $\xi \to -\infty$ (respectively as $\xi \to \infty$). Finally, we will use these pseudoexponential dichotomies to solve (6.6) and thereby prove Theorem 6.1.

6.1. **Spectral Gap.** The characteristic equation of $A(u,x,\rho)$ defined by (4.17) is

$$\mu^3 + \alpha\mu^2 + (x - 3u^2)\mu - \rho = 0. \tag{6.7}$$

Let $\rho = \theta + i\omega$ and $\mu = a + ib$, with $\theta,\omega,a,b \in \mathbb{R}$. Substituting these expressions into (6.7) yields

$$\left(a^3 - 3ab^2 + \alpha(a^2 - b^2) + a(x - 3u^2) - \theta\right) + i\left(3a^2b - b^3 + 2ab\alpha + b(x - 3u^2) - \omega\right) = 0. \tag{6.8}$$

From the real part of (6.8) we have $b^2 = \frac{a^3+\alpha a^2+a(x-3u^2)-\theta}{3a+\alpha}$. If $a > -\frac{\alpha}{3}$, then $a^3+\alpha a^2+a(x-3u^2)-\theta > 0$, so

$$\text{if } a > -\tfrac{\alpha}{3}, \text{ then } \theta < -a(-a^2-\alpha a+3u^2-x). \tag{6.9}$$

For $0 < \delta < \frac{\alpha^3}{192}$, let

$$I_1(\delta) = [a_1(\delta), b_1(\delta)] = \left[\frac{48\delta}{\alpha^2}, \frac{\alpha}{4}\right], \quad I_2(\delta) = [a_2(\delta), b_2(\delta)] = \left[-\frac{\alpha}{4}, -\frac{48\delta}{\alpha^2}\right].$$

Note that $b_1(\delta)$ and $a_2(\delta)$ are independent of δ.

Lemma 6.3. *Let $u \in \mathbb{R}$, let $0 < \delta < \frac{\alpha^3}{192}$, and let $\rho \in \mathbb{C}_\delta$. For $j = 1$ or 2, if $x \in J_j(u, \frac{\alpha^2}{3})$, then no eigenvalue of $A(u,x,\rho)$ has real part in $I_j(\delta)$.*

Proof. Under the assumptions of the lemma, let $x \in J_j(u, \frac{\alpha^2}{3})$, let $\rho = \theta + i\omega$, and let $\mu = a + ib \in I_j(\delta)$ satisfy (6.7). We will show that $\theta < -\delta$, which proves the result.

Using (6.9), for $j = 1$ we have

$$\delta = \frac{48\delta}{\alpha^2}\left(\frac{\alpha^2}{48}\right) \le a\left(\frac{\alpha^2}{48}\right) = a\left(-\frac{\alpha^2}{16} - \frac{\alpha^2}{4} + \frac{\alpha^2}{3}\right) < a(-a^2-\alpha a+3u^2-x) < -\theta.$$

For $j = 2$ we have

$$-\delta \ge a\left(\frac{\alpha^2}{48}\right) = -a\left(-\frac{\alpha^2}{48}\right) = -a\left(\frac{\alpha^2}{16} + \frac{\alpha^2}{4} - \frac{\alpha^2}{3}\right) > -a\left(-a^2-\alpha a+3u^2-x\right) > \theta.$$

□

Proposition 6.4. *Let $u \in \mathbb{R}$, let $0 < \delta < \frac{\alpha^3}{192}$, and let $\rho \in \mathbb{C}_\delta$. For $j = 1$ or 2, if $x \in J_j(u, \frac{\alpha^2}{3})$, then two eigenvalues of $A(u,x,\rho)$ have real part less than $a_j(\delta)$, and one eigenvalue of $A(u,x,\rho)$ has real part greater than $b_j(\delta)$.*

Proof. Assume the hypotheses of the proposition, and let $J_j = J_j(u, \frac{\alpha^2}{3})$. Note that $0 \in \mathbb{C}_\delta$. The eigenvalues of $A(u,x,0)$ are 0 and

$$\mu^\pm(u,x) = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 + 3u^2 - x}. \tag{6.10}$$

(Compare (2.9).)

If $x \in J_1$, then $\mu^- < 0$ and $\mu^+ > 0$. In fact, $\mu^+ > \frac{\alpha}{4}$, because

$$\begin{aligned} 3u^2 - x &> \frac{\alpha^2}{3} \\ \Rightarrow \frac{\alpha^2}{4} + 3u^2 - x &> \frac{7\alpha^2}{12} > \frac{9\alpha^2}{16} \\ \Rightarrow \sqrt{\frac{\alpha^2}{4} + 3u^2 - x} &> \frac{3\alpha}{4} \\ \Rightarrow -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + 3u^2 - x} &> \frac{\alpha}{4} \\ \Rightarrow \mu^+ &> \frac{\alpha}{4}. \end{aligned}$$

Hence, for $x \in J_1$, two eigenvalues of $A(u,x,0)$, μ^- and 0, have real part less than $a_1(\delta) = \frac{48\delta}{\alpha^2}$, and one, μ^+, has real part greater than $b_1(\delta) = \frac{\alpha}{4}$.

Now consider $x \in J_2$. The assumption $x > 3u^2 + \dfrac{\alpha^2}{3}$ implies that $\operatorname{Re}\mu^{\pm} = -\frac{\alpha}{2} < -\frac{\alpha}{4}$. Hence, for $x \in J_2$, two eigenvalues of $A(u,x,0)$, μ^- and μ^+, have real part less than $a_2(\delta) = -\frac{\alpha}{4}$, and one, 0, has real part greater than $b_2(\delta) = -\frac{48\delta}{\alpha^2}$.

We have shown that for $x \in J_j$, two eigenvalues of $A(u,x,0)$ have real part less than $a_j(\delta)$ and one has real part greater than $b_j(\delta)$. By Lemma 6.3 if $\rho \in \mathbb{C}_\delta$ and $x \in J_j$, then no eigenvalue of $A(u,x,\rho)$ has real part in $I_j(\delta)$. Since the eigenvalues depend continuously on (u,x,ρ), the proposition follows. □

6.2. **Distinct eigenvalues.** Let the eigenvalues of $A(u,x,\rho)$ be denoted μ_j, $j = 1, 2, 3$.

In (6.7) let $x = \frac{1}{q}$. Multipying by q yields

$$0 = q\mu^3 + \alpha q\mu^2 + (1 - 3u^2 q)\mu - \rho q = \mu - \rho q + \text{ higher order terms in } (q,\mu). \tag{6.11}$$

Therefore, for fixed (u,ρ), one of the eigenvalue is given by $\mu = \rho q + \mathcal{O}(q^2)$. We will let $\mu_3 = \rho q + \mathcal{O}(q^2)$. For fixed (u,ρ), $\mu_3 \to 0$ as $x \to \pm\infty$.

From (4.19) and (4.20) we have

$$\begin{aligned} \mu_1 + \mu_2 &= -\alpha - \mu_3 = -\alpha - \rho q + \mathcal{O}(q^2), \\ \mu_1\mu_2 &= \frac{\rho}{\mu_3} = \frac{1}{q + \mathcal{O}(q^2)}. \end{aligned}$$

Hence as $x \to \pm\infty$, $\mu_1 + \mu_2$ and $|\mu_1\mu_2|$ approach $-\alpha$ and ∞ respectively.

From (4.19) and (4.20) we also see that for $i = 1, 2$, $\frac{\rho}{\mu_i\mu_3} = -\alpha - \mu_i - \mu_3$. This implies that $0 = \mu_3\mu_i^2 + (\alpha + \mu_3)\mu_3\mu_i + \rho$. Hence

$$\mu_{1,2} = -\frac{\alpha + \rho q}{2} + \mathcal{O}(q^2) \pm \sqrt{\left(\frac{\alpha + \rho q}{2}\right)^2 + \mathcal{O}(q^2) - \frac{1}{q + \mathcal{O}(q^2)}} = \mathcal{O}(q^{-\frac{1}{2}}).$$

Examining the term under the square root sign, we see see that as $x = \frac{1}{q} \to -\infty$, μ_1 is real and approaches $-\infty$, μ_2 is real and approaches ∞. On the other hand, as $x = \frac{1}{q} \to \infty$, $\operatorname{Re}\mu_{1,2} \to -\frac{\alpha}{2}$, $\operatorname{Im}\mu_1 \to -\infty$, $\operatorname{Im}\mu_2 \to \infty$.

Lemma 6.5. *Let $u \in \mathbb{R}$.*

1. *If $\rho \in \mathbb{C}$, $m = m(u,\rho)$ is sufficiently large, and $x \in J_1(u,m)$, then the eigenvalues of $A(u,x,\rho)$ are distinct.*
2. *If $0 < \delta < \frac{\alpha^3}{192}$, $\rho \in \mathbb{C}_\delta$, and $x \in J_2(u,\frac{\alpha^2}{3})$, then the eigenvalues of $A(u,x,\rho)$ are distinct.*

Proof. For $x \in J_1(u,m)$ with m large, μ_3 is near 0, μ_1 is near $-\infty$, and μ_2 is near ∞.

For $x \in J_2(u,\frac{\alpha^2}{3})$, suppose $\mu_1 = \mu_2 \neq \mu_3$. From (4.19) and (4.20), $\rho = \mu_1^2\mu_3$ and $x - 3u^2 = -3\mu_1^2 - 2\alpha\mu_1$, so $\mu_1^2 + \frac{2\alpha}{3}\mu_1 + \frac{x-3u^2}{3} = 0$. Hence there are two possibilities for $\mu_1 = \mu_2$, namely $-\dfrac{\alpha}{3} \pm \sqrt{\left(\dfrac{\alpha}{3}\right)^2 + \dfrac{3u^2 - x}{3}}$. The corresponding possibilities for μ_3 are

$$-\alpha - 2\mu_1 = -\frac{\alpha}{3} \mp 2\sqrt{\left(\frac{\alpha}{3}\right)^2 + \frac{3u^2 - x}{3}}.$$

Since $x \in J_2(u,\frac{\alpha^2}{3})$, we have $x \geq 3u^2 + \frac{\alpha^2}{3}$, so $0 \geq \left(\frac{\alpha}{3}\right)^2 + \frac{3u^2 - x}{3}$. Therefore $\operatorname{Re}\mu_k = -\frac{\alpha}{3}$ for $k = 1, 2, 3$. This contradicts Proposition 6.4. □

6.3. **Change of basis and projections.** Let $u \in \mathbb{R}$, let $0 < \delta < \frac{\alpha^3}{192}$, and let $\rho \in \mathbb{C}_\delta$. Let $m = m(u,\rho) > \frac{\alpha^2}{3}$ be large enough so that Lemma 6.5 (1) applies. Let $x \in J_1(u,m)$ or $J_2(u,\frac{\alpha^2}{3})$. Then the eigenvectors of $A(u,x,\rho)$ are

$$\left\{ \begin{pmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \mu_3 \\ \mu_3^2 \end{pmatrix} \right\}.$$

We have numbered the eigenvalues so that for (u,ρ) fixed with $\rho \in \mathbb{C}_\delta$, for $x \sim -\infty$ in J_1, $\operatorname{Re}\mu_1 < \operatorname{Re}\mu_3 < a_1(\delta) < b_1(\delta) < \operatorname{Re}\mu_2$. For $x \sim \infty$ in J_2, $\operatorname{Re}\mu_1 < \operatorname{Re}\mu_2 < a_2(\delta) < b_2(\delta) < \operatorname{Re}\mu_3$. For $x \in J_j$, the pseudostable space is spanned by the eigenvectors associated with the eigenvalues of $A(u,x,\rho)$ with real part less than $a_j(\delta)$; the pseudounstable space is spanned by the eigenvector associated with the eigenvalue with real part greater than $b_j(\delta)$.

For $x \in J_1$, define

$$H(u,x,\rho) = \begin{pmatrix} \frac{1}{\mu_1^2} & 1 & \frac{1}{\mu_2^2} \\ \frac{1}{\mu_1} & \mu_3 & \frac{1}{\mu_2} \\ 1 & \mu_3^2 & 1 \end{pmatrix}.$$

Its inverse is

$$H^{-1}(u,x,\rho) = \begin{pmatrix} \frac{\mu_1^2\mu_2\mu_3}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\mu_1^2(\mu_2+\mu_3)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\mu_1\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\mu_1+\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \\ -\frac{\mu_1\mu_2^2\mu_3}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_2^2(\mu_1+\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix}.$$

For $x \in J_2$, define

$$H(u,x,\rho) = \begin{pmatrix} \frac{1}{\mu_1^2} & \frac{1}{\mu_2^2} & 1 \\ \frac{1}{\mu_1} & \frac{1}{\mu_2^2} & \mu_3 \\ 1 & 1 & \mu_3^2 \end{pmatrix}.$$

Its inverse is

$$H^{-1}(u,x,\rho) = \begin{pmatrix} \frac{\mu_1^2\mu_2\mu_3}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\mu_1^2(\mu_2+\mu_3)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ -\frac{\mu_1\mu_2^2\mu_3}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_2^2(\mu_1+\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ \frac{\mu_1\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\mu_1+\mu_2}{(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix}.$$

Let $P_2 = \operatorname{diag}(1,1,0)$. The projection onto the pseudostable space is $P(u,x,\rho) = HP_2H^{-1}$.

It is important to note that for fixed (u,ρ), $H^{-1}(u,x,\rho)$ and $P(u,x,\rho)$ are not bounded uniformly in x for $x \in J_j$. However, we do have the following result.

Proposition 6.6. *Let $u \in \mathbb{R}$, and let Ω be a compact subset of $\mathbb{C}$. Let $m > \frac{\alpha^3}{3}$ be sufficiently large so that Lemma 6.5 (1) applies to $J_1(u,m)$ for all $\rho \in \Omega$. Let $0 < \delta < \frac{\alpha^3}{192}$. Then there exists a constant $k = k(u,\Omega,m,\delta) > 0$ such that, if $x \in J_1(u,m)$ or $x \in J_2(u,\frac{\alpha^2}{3})$, and $\rho \in \mathbb{C}_\delta \cap \Omega$, then $\|P(u,x,\rho)(I-P_2)\| \le k$.*

Proof. Observe that as $x \to -\infty$,

$$H \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } H^{-1}(I-P_2) = \begin{pmatrix} 0 & 0 & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ 0 & 0 & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \\ 0 & 0 & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

As $x \to \infty$, we have

$$H \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } H^{-1}(I - P_2) = \begin{pmatrix} 0 & 0 & \frac{\mu_1^2}{(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ 0 & 0 & -\frac{\mu_2^2}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ 0 & 0 & \frac{1}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since both H and $H^{-1}(I - P_2)$ approach constant matrices, there exists k_1, $k_2 > 0$ such that $\|H\| \le k_1$ and $\|H^{-1}(I - P_2)\| \le k_2$. Therefore

$$\|P(u, x, \rho)(I - P_2)\| = \|HP_2H^{-1}(I - P_2)\| \le \|H\|\|P_2\|\|H^{-1}(I - P_2)\| \le \sqrt{2}k_1k_2.$$

□

6.4. **Pseudoexponential dichotomy.** Let $T(\xi, \zeta)$ be the family of state transition matrices for the linear system $Y_\xi = D(\xi)Y$. The system is said to have *pseudoexponential dichotomy* on J with spectral gap $(\hat{a}, \hat{b})$ if there exist $C > 0$ and projections $\mathcal{P}(\xi)$, $\xi \in J$, such that:

(i) P is continuous and uniformly bounded on J.
(ii) $T(\xi, \zeta)P(\zeta) = P(\xi)T(\xi, \zeta)$.
(iii) If $\xi > \zeta$, then $\|T(\xi, \zeta)P(\zeta)\| \le Ce^{\hat{a}(\xi-\zeta)}$.
(iv) If $\xi < \zeta$, then $\|T(\xi, \zeta)(I - P(\zeta))\| \le Ce^{\hat{b}(\xi-\zeta)}$.

Theorem 6.7 (Coppel's Roughness Theorem)**.** *Let C_0 and γ be positive numbers. Then there exist positive numbers ϵ_0 and L such that the following is true. Suppose $Y_\xi = D(\xi)Y$ has a pseudoexponential dichotomy on an interval J with projections $P(\xi)$. Suppose the pseudoexponential dichotomy has constants $C > 0$ and $\hat{a} < \hat{b}$ so that $C < C_0$ and $\hat{b} - \hat{a} > \gamma$. Let $0 < \epsilon < \epsilon_0$. If $\|E(\xi)\| < \epsilon$ for all $\xi \in J$, then the linear differential equation $Y_\xi = (D(\xi) + E(\xi))Y$ has a pseudoexponential dichotomy on J with projections $\tilde{P}(\xi)$, constant $\tilde{C}$, and exponents $\tilde{a}$ and $\tilde{b}$ with $\tilde{a} < \tilde{b}$ such that*

$$\|\tilde{P}(\xi) - P(\xi)\| < \epsilon L \text{ for all } \xi \in J, \quad |\tilde{C} - C| < \epsilon L, \quad |\tilde{a} - \hat{a}| < \epsilon L, \quad |\tilde{b} - \hat{b}| < \epsilon L.$$

6.5. **Persistence of pseudoexponential dichotomy.**

Theorem 6.8. *Let $\alpha > 0$, let (u^ℓ, u^r) satisfy the conditions of Theorem 2.3 or 2.4, and let Ω be a compact subset of $\mathbb{C}$. Let $m > \frac{\alpha^3}{3}$ be sufficiently large so that Lemma 6.5 (1) applies to $J_1(u^\ell, m)$ for all $\rho \in \Omega$. Let $0 < \delta_1 < \frac{\alpha^3}{192}$. Then there exist constants $\epsilon_1 > 0$ and $L \ge 1$ such that the following is true. Let $0 < \delta < \delta_1$, let $\rho \in \mathbb{C}_\delta \cap \Omega$, and let $0 < \epsilon < \epsilon_1$.*

(1) *Let $J_1 = J_1(u^\ell, m)$. Then*

$$Z_\xi = H^{-1}(u^\ell, \epsilon\xi, \rho)\hat{B}(\rho, \epsilon, \xi)H(u^\ell, \epsilon\xi, \rho)Z$$

has a pseudoexponential dichotomy on J_1^ϵ.

(2) *Let $J_2 = J_2(u^r, \frac{\alpha^2}{3})$. Then*

$$Z_\xi = H^{-1}(u^r, \epsilon\xi, \rho)\hat{B}(\rho, \epsilon, \xi)H(u^r, \epsilon\xi, \rho)Z$$

has a pseudoexponential dichotomy on J_2^ϵ.

In both cases the projections are near P_2, the constant C is near 1, and the exponents $\tilde{a}_j < \tilde{b}_j$ satisfy $|\tilde{a}_j - a_j(\delta)| < \epsilon L$ and $|\tilde{b}_j - b_j(\delta)| < \epsilon L$.

In order to prove this theorem, we shall first prove the following lemma.

Lemma 6.9. *Let $u^* \in \mathbb{R}$ and let Ω be a compact subset of $\mathbb{C}$. Let $m > \frac{\alpha^3}{3}$ be sufficiently large so that Lemma 6.5 (1) applies to $J_1(u^*, m)$ for all $\rho \in \Omega$. Let $J_1 = J_1(u^*, m)$ and $J_2 = J_2(u^*, \frac{\alpha^2}{3})$. Let $0 < \delta < \frac{\alpha^3}{192}$. Then for any $\rho \in \mathbb{C}_\delta \cap \Omega$ and any $\epsilon > 0$, the system*

$$Z_\xi = H^{-1}(u^*, \epsilon\xi, \rho)A(u^*, \epsilon\xi, \rho)H(u^*, \epsilon\xi, \rho)Z$$

has pseudoexponential dichotomies on both J_j^ϵ. For each dichotomy, the projection is P_2, the constant C is 1, and the constants $\hat{a}$ and $\hat{b}$ are respectively $a_j(\delta)$ and $b_j(\delta)$.

Proof. Let $N(u^*, x, \rho) = H^{-1}(u^*, x, \rho)A(u^*, x, \rho)H(u^*, x, \rho)$, so that

$$N(u^*, x, \rho) = \begin{pmatrix} \mu_1(u^*, x, \rho) & 0 & 0 \\ 0 & \mu_3(u^*, x, \rho) & 0 \\ 0 & 0 & \mu_2(u^*, x, \rho) \end{pmatrix} \text{ on } J_1,$$
$$N(u^*, x, \rho) = \begin{pmatrix} \mu_1(u^*, x, \rho) & 0 & 0 \\ 0 & \mu_2(u^*, x, \rho) & 0 \\ 0 & 0 & \mu_3(u^*, x, \rho) \end{pmatrix} \text{ on } J_2.$$

Consider the system

$$Z_\xi = N(u^*, \epsilon\xi, \rho)Z, \quad \epsilon\xi \in J_j. \tag{6.12}$$

For a given (u^*, ρ), the state transition matrix for (6.12) on J_1^ϵ is

$$S^\epsilon(\xi, \xi_0) = \text{diag}\left(e^{\int_{\xi_0}^{\xi} \mu_1(u^*, \epsilon\tau, \rho)d\tau}, e^{\int_{\xi_0}^{\xi} \mu_3(u^*, \epsilon\tau, \rho)d\tau}, e^{\int_{\xi_0}^{\xi} \mu_2(u^*, \epsilon\tau)d\tau, \rho}\right).$$

The conclusions of the lemma for J_1^ϵ follow easily. The argument for J_2^ϵ is analogous. ☐

To prove the theorem recall that $\hat{B}(\rho, \epsilon, \xi) = A(u^*, \epsilon\xi, \rho) + R(u^*, u^\epsilon(\xi), v^\epsilon(\xi))$. Under the change of variables

$$Y(\xi) = H(u^*, \epsilon\xi, \rho)Z(\xi), \tag{6.13}$$

the system (6.5) becomes

$$Z_\xi = (H^{-1}Y)_\xi = (H^{-1}AH + H^{-1}R(u^*, \epsilon, \xi)H - H^{-1}H_\xi)Z, \tag{6.14}$$

with H^{-1}, A, H, and $H_\xi = \epsilon H_x$ evaluated at $(u^*, \epsilon\xi, \rho)$.

Writing $\mu_j' = \frac{\partial \mu_j}{\partial x}$, we have

$$H_x = \begin{pmatrix} -\frac{2\mu_1'}{\mu_1^3} & 0 & -\frac{2\mu_2'}{\mu_2^3} \\ -\frac{\mu_1'}{\mu_1^2} & \mu_3' & -\frac{\mu_2'}{\mu_2^2} \\ 0 & 2\mu_3\mu_3' & 0 \end{pmatrix} \text{ on } J_1, \text{ and } H_x = \begin{pmatrix} -\frac{2\mu_1'}{\mu_1^3} & -\frac{2\mu_2'}{\mu_2^3} & 0 \\ -\frac{\mu_1'}{\mu_1^2} & -\frac{\mu_2'}{\mu_2^2} & \mu_3' \\ 0 & 0 & 2\mu_3\mu_3' \end{pmatrix} \text{ on } J_2.$$

Now $\frac{\partial \mu_j}{\partial x} = \frac{\partial \mu_j}{\partial q}\frac{dq}{dx} = -q^2\frac{\partial \mu_j}{\partial q}$. Therefore

$$\begin{aligned} \frac{\partial \mu_{1,2}}{\partial x} &= -q^2\Bigg(-\frac{\rho}{2} + \mathcal{O}(q) \pm \\ \Bigg(\frac{\rho(\alpha + q\rho)}{4} &+ \mathcal{O}(q) + \frac{1}{2q^2(1 + \mathcal{O}(q))}\Bigg)\Bigg(\left(\frac{\alpha + q\rho}{2}\right)^2 + \mathcal{O}(q^2) - \frac{1}{q + \mathcal{O}(q^2)}\Bigg)^{-\frac{1}{2}}\Bigg) \\ &= -q^2\left(\mathcal{O}(q^0) \pm \mathcal{O}(q^{-2})\left(\mathcal{O}(q^{-1})\right)^{-\frac{1}{2}}\right) = -q^2\mathcal{O}\left(q^{-\frac{3}{2}}\right) = \mathcal{O}(q^{\frac{1}{2}}), \end{aligned}$$

$$\frac{\partial \mu_3}{\partial x} = -q^2(\rho + \mathcal{O}(q)) = \mathcal{O}(q^2).$$

Hence as $x \to \pm\infty$, H_x approaches the zero matrix. This implies that there exists k_3, such that $\|H_x\| < k_3$ on J_j.

Now H^{-1} is not bounded on J_j. However, on J_1,

$$H^{-1}H_x = \begin{pmatrix} \frac{\mu_1'(-2\mu_3\mu_2+\mu_1(\mu_2+\mu_3))}{\mu_1(\mu_2-\mu_1)(\mu_3-\mu_1)} & \frac{\mu_3'\mu_1^2(\mu_3-\mu_2)}{(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_2'\mu_1^2(\mu_2-\mu_3)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\mu_1'(\mu_1-\mu_2)}{(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{-\mu_3'(\mu_1+\mu_2-2\mu_3)}{(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{\mu_2'(\mu_2-\mu_1)}{\mu_2^2(\mu_3-\mu_1)(\mu_3-\mu_2)} \\ \frac{\mu_1'\mu_2^2(\mu_3-\mu_1)}{\mu_1^2(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_3'\mu_2^2(\mu_1-\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{-\mu_2'(-2\mu_1\mu_3+\mu_2(\mu_3+\mu_1))}{\mu_2(\mu_3-\mu_2)(\mu_2-\mu_1)} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{O}(q) & \mathcal{O}(q^{\frac{2}{3}}) & \mathcal{O}(q) \\ \mathcal{O}(q) & \mathcal{O}(q^{\frac{5}{2}}) & \mathcal{O}(q^2) \\ \mathcal{O}(q) & \mathcal{O}(q^{\frac{3}{2}}) & \mathcal{O}(q) \end{pmatrix}$$

is bounded. Similarly, on J_2,

$$H^{-1}H_x = \begin{pmatrix} \frac{\mu_1'(-2\mu_3\mu_2+\mu_1\mu_2+\mu_1\mu_3)}{\mu_1(\mu_3-\mu_1)(\mu_2-\mu_1)} & -\frac{\mu_2'\mu_1^2(\mu_3-\mu_2)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} & \frac{\mu_3'\mu_1^2(\mu_3-\mu_2)}{\mu_2^2(\mu_3-\mu_1)(\mu_2-\mu_1)} \\ \frac{\mu_1'\mu_2^2(\mu_3-\mu_1)}{\mu_1^2(\mu_3-\mu_1)(\mu_3-\mu_2)} & \frac{\mu_2'(2\mu_1\mu_3-\mu_1\mu_2-\mu_2\mu_3)}{\mu_2(\mu_3-\mu_2)(\mu_2-\mu_1)} & \frac{\mu_3'\mu_2^2(\mu_1-\mu_3)}{(\mu_3-\mu_2)(\mu_2-\mu_1)} \\ \frac{\mu_1'(\mu_1-\mu_2)}{\mu_1^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & \frac{\mu_2'(\mu_2-\mu_1)}{\mu_2^2(\mu_3-\mu_2)(\mu_3-\mu_1)} & -\frac{\mu_3'(\mu_1+\mu_2-2\mu_3))}{(\mu_3-\mu_2)(\mu_3-\mu_1)} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{O}(q) & \mathcal{O}(q) & \mathcal{O}(q^{\frac{5}{2}}) \\ \mathcal{O}(q) & \mathcal{O}(q^{\frac{5}{2}}) & \mathcal{O}(q^{\frac{3}{2}}) \\ \mathcal{O}(q) & \mathcal{O}(q^2) & \mathcal{O}(q^{\frac{5}{2}}) \end{pmatrix}$$

is bounded. Therefore $H^{-1}H_\xi = \epsilon H_x$ is $\mathcal{O}(\epsilon)$ on each of these intervals.

On J_1, we shall use $u^* = u^\ell$, so

$$H^{-1}RH = H^{-1}(u^\ell, x, \rho)R(u^\ell, \hat{u}^\epsilon(x), \hat{v}^\epsilon(x))H(u^\ell, x, \rho) \text{ with } x = \epsilon\xi.$$

On J_2, we shall use $u^* = u^r$, so

$$H^{-1}RH = H^{-1}(u^r, x, \rho)R(u^r, \hat{u}^\epsilon(x), \hat{v}^\epsilon(x))H(u^r, x, \rho) \text{ with } x = \epsilon\xi.$$

On J_j, $R(u^r, \hat{u}^\epsilon(x), \hat{v}^\epsilon(x))$ is of order ϵ uniformly in x, and for fixed small ϵ it decreases exponentially as $x \to \pm\infty$ at a rate independent of ϵ. Although H^{-1} increases algebraically on J_j as $x \to \pm\infty$, the product $H^{-1}R$ remains of order ϵ uniformly in x. Therefore on J_j, $H^{-1}RH$ is $\mathcal{O}(\epsilon)$ uniformly in x.

Since $H^{-1}H_\xi$ and $H^{-1}RH$ are $\mathcal{O}(\epsilon)$ on J_j uniformly in x, the theorem follows from Lemma 6.9 and Coppel's Roughness Theorem.

6.6. **Notation.** For the system (6.14), with $u^* = u^\ell$ on J_1^ϵ and $u^* = u^r$ on J_2^ϵ, let $\Phi_j(\rho, \epsilon, \xi, \zeta)$ denote the family of state transition matrices on J_j^ϵ, and let $\tilde{Q}_j(\rho, \epsilon, \xi)$ denote the projection for the pseudoexponential dichotomy on J_j^ϵ. We will usually suppress ρ and ϵ, and just write $\Phi_j(\xi, \zeta)$ and $\tilde{Q}_j(\xi)$. Similarly, we will use $H_1(\xi)$ to denote $H(u^\ell, \epsilon\xi, \rho)$ on J_1^ϵ, and $H_2(\xi)$ to denote $H(u^r, \epsilon\xi, \rho)$ on J_2^ϵ.

For small $\delta > 0$, there exists constants $C > 0$ and $a_j(\delta) < b_j(\delta)$ so that on J_j^ϵ, for $\xi > \zeta$,

$$\|\Phi_j(\xi, \zeta)\tilde{Q}_j(\zeta)\| \leq C e^{a_j(\delta)(\xi - \zeta)}, \tag{6.15}$$

and for $\xi < \zeta$,

$$\|\Phi_j(\xi,\zeta)(I - \tilde{Q}_j(\zeta))\| \le Ce^{b_j(\delta)(\xi-\zeta)}. \tag{6.16}$$

Since $Y(\zeta) = H_j(\zeta)Z(\zeta)$ (see (6.13)), the family of state transition matrices for (6.5) on J_j^ϵ is

$$Y(\xi) = T(\xi,\zeta)Y(\zeta) = H_j(\xi)\Phi_j(\xi,\zeta)H_j^{-1}(\zeta)Y(\zeta). \tag{6.17}$$

The system (6.5) on J_j^ϵ has a pseudoexponential dichotomy with projections

$$Q_j(\rho,\epsilon,\xi) = Q_j(\xi) = H_j(\xi)\tilde{Q}_j(\xi)H_j^{-1}(\xi). \tag{6.18}$$

We have $T(\xi,\zeta)Q_j(\zeta) = Q_j(\xi)T(\xi,\zeta)$.

6.7. **Proof of Theorem 6.1.** Fix $\rho \in \mathbb{C}_\delta \cap \Omega$. Notice that $\delta = \frac{\epsilon}{L} < \delta_0 \le \delta_1 < \frac{\alpha^3}{192}$ and $\epsilon < L\delta_0 \le \epsilon_0 \le \epsilon_1$. Hence by Theorem 6.8, (6.5) has pseudoexponential dichotomies on J_1^ϵ and J_2^ϵ, with projections $Q_j(\xi) = Q_j(\rho,\epsilon,\xi)$ as in the previous section. The dichotomies can be extended to the intervals $(-\infty,0]$ and $[0,\infty)$. The constants C in the extended dichotomies may increase (in fact may approach infinity) as $\epsilon \to 0$, but the exponents in the extended dichotomies do not change. The constants C are not important to our results, so we ignore the change.

Let $\Lambda = R(Q_2(0)) \cap R(I - Q_1(0))$. Since $R(Q_2(0))$ has dim 2 and $R(I - Q_1(0))$ has dim 1, Λ has dim 0 or 1. First suppose Λ has dim 1. If $Y_0 \in \Lambda$, let $Y(\xi) = (U(\xi), V(\xi), W(\xi))$ be the solution of $Y_\xi = B(\rho,\epsilon,\xi)Y$ with $Y(0) = Y_0$. Then $U(\xi)$ is an eigenfunction of $\mathcal{T}$ for the eigenvalue ρ, and all eigenfunctions arise in this way. Since Λ has dim 1, ρ is an eigenvalue of $\mathcal{T}$ of geometric multiplicity one.

Now suppose $\Lambda = \{0\}$. Since $R(Q_2(0))$ has dimension 2 and $R(I - Q_1(0))$ has dimension 1, we have $R(Q_2(0)) \oplus R(I - Q_1(0)) = \mathbb{R}^3$. Let $f \in C(\epsilon\gamma, \mathbb{R})$. Assume that $Y(\xi)$ is a solution to (6.6) in $C(\epsilon\gamma, \mathbb{R})$. Let $Y_s = Q_2(0)Y(0)$, and let $Y_u = (I - Q_1(0))Y(0)$.

Let $0 \le \xi < \infty$ and $\tau > \xi$. Using the variation of parameters formula, we can write

$$\begin{aligned} Y(\xi) &= Q_2(\xi)Y(\xi) + (I - Q_2(\xi))Y(\xi) \\ &= Q_2(\xi)T(\xi,0)Y(0) + \int_0^\xi Q_2(\xi)T(\xi,\zeta)F(\zeta)d\zeta + (I - Q_2(\xi))T(\xi,\tau)Y(\tau) \\ &\quad + \int_\tau^\xi (I - Q_2(\xi))T(\xi,\zeta)F(\zeta)d\zeta. \end{aligned} \tag{6.19}$$

We claim that as $\tau \to \infty$, the term $(I - Q_2(\xi))T(\xi,\tau)Y(\tau) \to 0$. Based on the fact that (6.14) has a pseudoexponential dichotomy on J_2^ϵ, and on the equations (6.17), (6.18), we have

$$\begin{aligned} \|(I - Q_2(\xi))T(\xi,\tau)Y(\tau)\| &= \|T(\xi,\tau)(I - Q_2(\xi))Y(\tau)\| \\ &= \|H_2(\xi)\Phi_2(\xi,\tau)H_2^{-1}(\tau)(I - H_2(\tau)\tilde{Q}_2(\tau)H_2^{-1}(\tau))Y(\tau)\| \\ &= \|H_2(\xi)\Phi_2(\xi,\tau)H_2^{-1}(\tau)\Big(H_2(\tau)(I - \tilde{Q}_2(\tau))H_2^{-1}(\tau)\Big)Y(\tau)\| \\ &= \|H_2(\xi)\Phi_2(\xi,\tau)(I - \tilde{Q}_2(\tau))H_2^{-1}(\tau)Y(\tau)\| \\ &\le \|H_2(\xi)\|\|\Phi_2(\xi,\tau)(I - \tilde{Q}_2(\tau))\|\|H_2^{-1}(\tau)\|\|Y(\tau)\| \end{aligned}$$

Recall that for $\xi \leq \tau$, $\|\Phi_2(\xi,\tau)(I-\tilde{Q}_2(\tau)\| \leq Ce^{\tilde{b}_2(\xi-\tau)}$ with $\tilde{b}_2$ given in the statement of Theorem 6.8. Then

$$\begin{aligned} \epsilon\gamma + \tilde{b}_2 > \epsilon\gamma + b_2 + L\epsilon &= \epsilon\gamma - \frac{48\delta}{\alpha^2} + L\epsilon \\ &= \epsilon\gamma - \frac{48\epsilon}{L\alpha^2} + L\epsilon \\ &= \epsilon\left(\gamma - \left(\frac{48}{L\alpha^2} - L\right)\right) > 0. \end{aligned}$$

Hence,

$$\begin{aligned} \|(I-Q_2(\xi))T(\xi,\tau)Y(\tau)\| &\leq \|H_2(\xi)\|\|\Phi_2(\xi,\tau)(I-\tilde{Q}_2(\tau))\|\|H_2^{-1}(\tau)\|\|Y(\tau)\| \\ &\leq \tilde{D}e^{\tilde{b}_2(\xi-\tau)-\epsilon\gamma\tau}\|H_2^{-1}(\tau)\|\|Y(\tau)\| \\ &\leq \tilde{D}e^{-(\epsilon\gamma+\tilde{b}_2)\tau}\|H_2^{-1}(\tau)\|\|Y(\tau)\| \end{aligned}$$

Since $\|H_2^{-1}(\tau)\|$ grows algebraically as $\tau \to \infty$, the decaying exponential term will dominate the behavior of the product as $\tau \to \infty$. Thus as $\tau \to \infty$, $\|(I-Q_2(\xi))T(\xi,\tau)Y(\tau)\| \to 0$.

We can therefore rewrite (6.19) as

$$\begin{aligned} Y(\xi) &= T(\xi,0)Q_2(0)Y(0) + \int_0^\xi T(\xi,\zeta)Q_2(\zeta)F(\zeta)d\zeta + \int_\infty^\xi T(\xi,\zeta)(I-Q_2(\zeta))F(\zeta)d\zeta \\ &= T(\xi,0)Y_s + \int_0^\xi T(\xi,\zeta)Q_2(\zeta)F(\zeta)d\zeta + \int_\infty^\xi T(\xi,\zeta)(I-Q_2(\zeta))F(\zeta)d\zeta. \end{aligned}$$

Similarly, let $-\infty < \xi \leq 0$ and $\tau < \xi$. Using the variation of parameters formula, we can write

$$\begin{aligned} Y(\xi) &= Q_1(\xi)Y(\xi) + (I-Q_1(\xi))Y(\xi) \\ &= Q_1(\xi)T(\xi,\tau)Y(\tau) + \int_\tau^\xi Q_1(\xi)T(\xi,\zeta)F(\zeta)d\zeta + (I-Q_1(\xi))T(\xi,0)Y(0) \\ &\quad + \int_0^\xi (I-Q_1(\xi))T(\xi,\zeta)F(\zeta)d\zeta. \end{aligned} \tag{6.20}$$

As $\tau \to -\infty$, $\|Q_1(\xi)T(\xi,\tau)Y(\tau)\| \to 0$. This can be seen by observing that

$$\begin{aligned} \|Q_1(\xi)T(\xi,\tau)Y(\tau)\| &= \|T(\xi,\tau)Q_1(\tau)Y(\tau)\| \\ &= \|H_1(\xi)\Phi_1(\xi,\tau)H_1^{-1}(\tau)H_1(\tau)\tilde{Q}_1(\tau)H_1^{-1}(\tau)Y(\tau)\| \\ &\leq \|H_1(\xi)\|\|\Phi_1(\xi,\tau)\tilde{Q}_1(\tau)\|\|H_1^{-1}(\tau)\|\|Y(\tau)\|. \end{aligned}$$

Recall that for $\xi \geq \tau$, $\|\Phi_1(\xi,\tau)\tilde{Q}_1(\tau)\| \leq Ce^{\tilde{a}_1(\xi-\tau)}$ with $\tilde{a}_1$ given in the statement of Theorem 6.8. Then

$$\begin{aligned} \epsilon\gamma - \tilde{a}_1 > \epsilon\gamma - (a_1 + L\epsilon) &= \epsilon\gamma - \frac{48\delta}{\alpha^2} - L\epsilon \\ &= \epsilon\gamma - \frac{48\epsilon}{L\alpha^2} - L\epsilon \\ &= \epsilon\left(\gamma - \left(\frac{48}{L\alpha^2} + L\right)\right) > 0. \end{aligned}$$

So we have,

$$\begin{aligned}\|Q_1(\xi)T(\xi,\tau)Y(\tau)\| &\leq \|H_1(\xi)\|\|\Phi_1(\xi,\tau)\tilde{Q}_1(\tau)\|\|H_1^{-1}(\tau)\|\|Y(\tau)\| \\ &\leq \tilde{C}e^{(\tilde{a}_1(\xi-\tau)-\epsilon\gamma|\tau|)}\|H_1^{-1}(\tau)\|\|Y(\tau)\|_{\epsilon\gamma} \\ &\leq \tilde{C}e^{(\epsilon\gamma-\tilde{a}_1)\tau}\|H_1^{-1}(\tau)\|\|Y(\tau)\|_{\epsilon\gamma}.\end{aligned}$$

Since $\|H_1^{-1}(\tau)\|$ grows algebraically as $\tau \to -\infty$, the decaying exponential term will dominate the behavior of the product as $\tau \to -\infty$. Therefore $\|Q_1(\xi)T(\xi,\tau)Y(\tau)\| \to 0$ as $\tau \to -\infty$.

We can therefore rewrite (6.20) as

$$\begin{aligned}Y(\xi) &= \int_{-\infty}^{\xi} T(\xi,\zeta)Q_1(\zeta)F(\zeta)d\zeta + T(\xi,0)(I-Q_1(0))Y(0) \\ &\quad + \int_0^{\xi} T(\xi,\zeta)(I-Q_1(\zeta))F(\zeta)d\zeta \\ &= \int_{-\infty}^{\xi} T(\xi,\zeta)Q_1(\zeta)F(\zeta)d\zeta + T(\xi,0)Y_u + \int_0^{\xi} T(\xi,\zeta)(I-Q_1(\zeta))F(\zeta)d\zeta.\end{aligned}$$

We conclude that if equation (6.6), with $f \in C(\epsilon\gamma,\mathbb{R}_\xi)$, has a solution $Y \in C(\epsilon\gamma,\mathbb{R}_\xi)$, then Y must be given by expression

$$\int_{-\infty}^{\xi} T(\xi,\zeta)Q_1(\zeta)F(\zeta)d\zeta + T(\xi,0)Y_u + \int_0^{\xi} T(\xi,\zeta)(I-Q_1(\zeta))F(\zeta)d\zeta$$

for $\xi \leq 0$ and the expression

$$T(\xi,0)Y_s + \int_0^{\xi} T(\xi,\zeta)Q_2(\zeta)F(\zeta)d\zeta + \int_{\infty}^{\xi} T(\xi,\zeta)(I-Q_2(\zeta))F(\zeta)d\zeta$$

for $\xi \geq 0$.

It is easy to check that these formulas define solutions of (6.6) on $\xi \leq 0$ and $\xi \geq 0$ respectively. If $Y \in C(\epsilon\gamma,\mathbb{R}_\xi)$, then it must be continuous at 0. We will use this fact to solve for Y_u and Y_s. In addition, we will show that the $\|Y(\xi)\|_{\epsilon\gamma} \leq K\|f\|_{\epsilon\gamma}$, for some constant K.

Using the fact that $(I-P_2)F = F$, we observe when $\xi \geq 0$, the norm of $Y(\xi)$ is bounded as follows:

$$\begin{aligned}\|Y(\xi)\| &\leq \|T(\xi,0)Y_s\| + \int_0^{\xi} \|T(\xi,\zeta)Q_2(\zeta)(I-P_2)F(\zeta)\|d\zeta \\ &\quad + \int_{\xi}^{\infty} \|T(\xi,\zeta)(I-Q_2(\zeta))(I-P_2)F(\zeta)d\zeta\| \\ &= \|H_2(\xi)\Phi_2(\xi,0)H_2^{-1}(0)Y_s\| \\ &\quad + \int_0^{\xi} \|H_2(\xi)\Phi_2(\xi,\zeta)H_2^{-1}(\zeta)H_2(\zeta)\tilde{Q}_2(\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\ &\quad + \int_{\xi}^{\infty} \|H_2(\xi)\Phi_2(\xi,\zeta)H_2^{-1}(\zeta)H_2(\zeta)(I-\tilde{Q}_2(\zeta))H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\ &= \|H_2(\xi)\Phi_2(\xi,0)H_2^{-1}(0)Y_s\| + \int_0^{\xi} \|H_2(\xi)\Phi_2(\xi,\zeta)\tilde{Q}_2(\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\ &\quad + \int_{\xi}^{\infty} \|H_2(\xi)\Phi_2(\xi,\zeta)(I-\tilde{Q}_2(\zeta))H_2^{-1}(\zeta)(I-P_2)F(\zeta)d\zeta\|\end{aligned}$$

$$
\begin{aligned}
&\le \|H_2(\xi)\|\|\Phi_2(\xi,0)\|\|H_2^{-1}(0)\|\|Y_s\| \\
&\quad + \int_0^\xi \|H_2(\xi)\|\Phi_2(\xi,\zeta)\tilde{Q}_2(\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\quad + \int_\xi^\infty \|H_2(\xi)\|\|\Phi_2(\xi,\zeta)(I-\tilde{Q}_2(\zeta))\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)d\zeta\| \\
&\le \hat{D}_1 e^{\tilde{a}_2\xi}\|Y_s\| + \hat{D}_2\|F\|_{\epsilon\gamma}\int_0^\xi e^{\tilde{a}_2(\xi-\zeta)-\epsilon\gamma\zeta}d\zeta + \hat{D}_3\|F\|_{\epsilon\gamma}\int_\xi^\infty e^{\tilde{b}_2(\xi-\zeta)-\epsilon\gamma\zeta}d\zeta \\
&\le \hat{D}_1 e^{\tilde{a}_2\xi}\|Y_s\| + \hat{D}_4\|F\|_{\epsilon\gamma}(e^{-\epsilon\gamma\xi}-e^{\tilde{a}_2\xi}) + \hat{D}_5\|F\|_{\epsilon\gamma}e^{-\epsilon\gamma\xi}.
\end{aligned}
$$

Multiplying $\|Y(\xi)\|$ by $e^{\epsilon\gamma\xi}$, gives $e^{\epsilon\gamma\xi}\|Y(\xi)\| \le C_2\|Y_s\| + k_2\|F\|_{\epsilon\gamma}$.

Again using the fact that $(I-P_2)F = F$, we observe when $\xi \le 0$, the norm of $Y(\xi)$ is bounded as follows:

$$
\begin{aligned}
\|Y(\xi)\| &\le \int_{-\infty}^\xi \|T(\xi,\zeta)Q_1(\zeta)F(\zeta)\|d\zeta + \|T(\xi,0)Y_u\| + \int_\xi^0 \|T(\xi,\zeta)(I-Q_1(\zeta))F(\zeta)\|d\zeta \\
&= \int_{-\infty}^\xi \|H_1(\xi)\Phi_1(\xi,\zeta)H_1^{-1}(\zeta)H_1(\zeta)\tilde{Q}_1(\zeta)H^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \|H_1(\xi)\Phi_1(\xi,0)H_1^{-1}(0)Y_u\| \\
&\quad + \int_\xi^0 \|H_1(\xi)\Phi_1(\xi,\zeta)H_1^{-1}(\zeta)H_1(\zeta)(I-\tilde{Q}_1(\zeta))H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^\xi \|H_1(\xi)\Phi_1(\xi,\zeta)\tilde{Q}_1(\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta + \|H_1(\xi)\Phi_1(\xi,0)H_1^{-1}(0)Y_u\| \\
&\quad + \int_\xi^0 \|H_1(\xi)\Phi_1(\xi,\zeta)(I-\tilde{Q}_1(\zeta))H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\le \int_{-\infty}^\xi \|H_1(\xi)\|\|\Phi_1(\xi,\zeta)\tilde{Q}_1(\zeta)\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\quad + \|H_1(\xi)\|\|\Phi_1(\xi,0)\|\|H_1^{-1}(0)\|\|Y_u\| \\
&\quad + \int_\xi^0 \|H_1(\xi)\|\|\Phi_1(\xi,\zeta)(I-\tilde{Q}_1(\zeta))\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\le \hat{C}_1\|F\|_{\epsilon\gamma}\int_{-\infty}^\xi e^{\tilde{a}_1(\xi-\zeta)-\epsilon\gamma|\zeta|}d\zeta + \hat{C}_2 e^{\tilde{b}_1\xi}\|Y_u\| + \hat{C}_3\|F\|_{\epsilon\gamma}\int_\xi^0 e^{\tilde{b}_1(\xi-\zeta)-\epsilon\gamma|\zeta|}d\zeta \\
&= \hat{C}_4\|F\|_{\epsilon\gamma}e^{\epsilon\gamma\xi} + \hat{C}_2 e^{\tilde{b}_1\xi}\|Y_u\| + \hat{C}_5\|F\|_{\epsilon\gamma}(e^{\epsilon\gamma\xi}-e^{\tilde{b}_1\xi}).
\end{aligned}
$$

Multiplying by $e^{\epsilon\gamma|\xi|}$ gives $e^{\epsilon\gamma|\xi|}\|Y(\xi)\| \le C_1\|Y_s\| + k_1\|F\|_{\epsilon\gamma}$.

The values Y_s and Y_u are chosen so that $Y(\xi)$ is continuous at 0. So for $\xi = 0$, we have

$$
Y_s - Y_u = \int_{-\infty}^0 T(0,\zeta)Q_1(\zeta)F(\zeta)d\zeta + \int_0^\infty T(0,\zeta)(I-Q_2(\zeta))F(\zeta)d\zeta.
$$

Since $R(Q_2(0))$ and $R(I-Q_1(0))$ are complementary, we can define a projection $\hat{Q}$ on $\mathbb{R}^3$ with $R(\hat{Q}) = R(Q_2(0))$ and $R(I-\hat{Q}) = R(I-Q_1(0))$. Let $\hat{Y}$ denote the right hand side of the previous equation. Then $\hat{Q}\hat{Y} = Y_s$ and $(I-\hat{Q})\hat{Y} = -Y_u$. Therefore:

$$
\begin{aligned}
\|\hat{Q}\hat{Y}\| &\leq \int_{-\infty}^{0} \|\hat{Q}Q_1(0)T(0,\zeta)F(\zeta)\|d\zeta + \int_0^\infty \|\hat{Q}(I-Q_2(0))T(0,\zeta)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|\hat{Q}H_1(0)\tilde{Q}_1(0)H_1^{-1}(0)H_1(0)\Phi_1(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_0^\infty \|\hat{Q}H_2(0)(I-\tilde{Q}_2(0))H_2^{-1}(0)H_2(0)\Phi_2(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|\hat{Q}H_1(0)\tilde{Q}_1(0)\Phi_1(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_0^\infty \|\hat{Q}H_2(0)(I-\tilde{Q}_2(0))\Phi_2(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\leq \int_{-\infty}^{0} ||\hat{Q}\|\|H_1(0)\|\|\tilde{Q}_1(0)\Phi_1(0,\zeta)\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)|d\zeta \\
&\quad + \int_0^\infty \|\hat{Q}\|\|H_2(0)\|\|(I-\tilde{Q}_2(0))\Phi_2(0,\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\leq \hat{E}_1\|F\|_{\epsilon\gamma}\int_{-\infty}^{0} e^{-\tilde{a}_1\zeta-\epsilon\gamma|\zeta|}d\zeta + \hat{E}_2\|F\|_{\epsilon\gamma}\int_0^\infty e^{-\tilde{b}_2\zeta-\epsilon\gamma\zeta}d\zeta \\
&= E_1\|F\|_{\epsilon\gamma}
\end{aligned}
$$

$$
\begin{aligned}
\|(I-\hat{Q})\hat{Y}\| &\leq \int_{-\infty}^{0} \|(I-\hat{Q})Q_1(0)T(0,\zeta)F(\zeta)\|d\zeta + \int_0^\infty \|(I-\hat{Q})(I-Q_2(0))T(0,\zeta)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|(I-\hat{Q})H_1(0)\tilde{Q}_1(0)H_1^{-1}(0)H_1(0)\Phi_1(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_0^\infty \|(I-\hat{Q})H_2(0)(I-\tilde{Q}_2(0))H_2^{-1}(0)H_2(0)\Phi(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&= \int_{-\infty}^{0} \|(I-\hat{Q})H_1(0)\tilde{Q}_1(0)\Phi_1(0,\zeta)H_1^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\quad + \int_0^\infty \|(I-\hat{Q})H_2(0)(I-\tilde{Q}_2(0))\Phi_2(0,\zeta)H_2^{-1}(\zeta)(I-P_2)F(\zeta)\|d\zeta \\
&\leq \int_{-\infty}^{0} \|I-\hat{Q}\|\|H_1(0)\|\|\tilde{Q}_1(0)\Phi_1(0,\zeta)\|\|H_1^{-1}(\zeta)(I-P_2)\|\|F(\zeta)||d\zeta \\
&\quad + \int_0^\infty \|I-\hat{Q}\|\|H_2(0)\|\|(I-\tilde{Q}_2(0))\Phi_2(0,\zeta)\|\|H_2^{-1}(\zeta)(I-P_2)\|\|F(\zeta)\|d\zeta \\
&\leq \hat{E}_3\|F\|_{\epsilon\gamma}\int_{-\infty}^{0} e^{-\tilde{a}_1\zeta-\epsilon\gamma|\zeta|}d\zeta + \hat{E}_4\|F\|_{\epsilon\gamma}\int_0^\infty e^{-\tilde{b}_2\zeta-\epsilon\gamma\zeta}d\zeta \\
&= E_2\|F\|_{\epsilon\gamma}
\end{aligned}
$$

So we have, $\|Y_s\| \leq E_1\|F\|_{\epsilon\gamma}$ for $\xi \geq 0$ and $\|Y_u\| = \|-Y_u\| \leq E_2\|F\|_{\epsilon\gamma}$ for $\xi < 0$. This implies that, $\|Y(\xi)\|_{\epsilon\gamma} = e^{\epsilon\gamma\xi}|Y(\xi)|$ is bounded by a positive multiple of $\|F\|_{\epsilon\gamma}$. Therefore, $Y(\xi) \in C(\epsilon\gamma, \mathbb{R}_\xi)$ for all $\xi \in \mathbb{R}$. Since solving $(\mathcal{T}^\epsilon - \rho I)U = f$ is equivalent to solving (6.6), U is the first component of the solution of (6.6), which is $Y_1(\xi)$. Thus, there exists a constant

$K > 0$ so that $\|U\|_{\epsilon\gamma} \le K\|F\|_{\epsilon\gamma} = K\|f\|_{\epsilon\gamma}$. Therefore, $(\mathcal{T}^\epsilon - \rho I)^{-1}$ exists and is bounded on $C(\epsilon\gamma, \mathbb{R}_\xi)$.

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Appendix A. Normal hyperbolicity

In this section, for the system (2.17)–(2.20), we discuss the persistence, for small $\epsilon > 0$, of the stable and unstable manifolds of the manifolds $\mathcal{P}$ and $\mathcal{Q}$ defined in Subsection 2.4, and their invariant foliations. Then we shall briefly discuss the analogous issues for the manifolds $\mathcal{K}$ and $\mathcal{L}$ defined in Subsection 4.4, and the manifolds $\mathcal{M}_\epsilon$ and $\mathcal{N}_\epsilon$ defined in Subsection 5.1. In fact we shall only discuss $\mathcal{P}$, $\mathcal{K}$, and $\mathcal{M}_\epsilon$.

A.1. **The manifold $\mathcal{P}$.** The linearization of (2.21)–(2.24) at an equilibrium $(u, 0, 0, x)$ has the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3u^2 - x & -\alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues are 0, 0, and $\mu^\pm(u, x)$ given by (6.10). The double eigenvalue 0 has a 2-dimensional eigenspace spanned by $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ Eigenvectors corresponding to μ^- and μ^+ are respectively $((\mu^-)^{-2}, (\mu^-)^{-1}, 1, 0)$ and $((\mu^+)^{-2}, (\mu^+)^{-1}, 1, 0)$. For fixed u, as $x \to -\infty$, $\mu^- \to -\infty$ and $\mu^+ \to \infty$. The corresponding eigenvectors approach $(0, 0, 1, 0)$, i.e., they become less and less linearly independent. This is another way of saying that we have less and less normal hyperbolicity as $x \to -\infty$, even though the eigenvalues are real and far from 0 and each other.

Another choice of eigenvectors for μ^- and μ^+ is $((\mu^-)^{-1}, 1, \mu^-, 0)$ and $((\mu^+)^{-1}, 1, \mu^+, 0)$. For $x < 0$, let $x = -y^{-2}$, $y < 0$, so that $y = -(-x)^{-\frac{1}{2}}$. Then for x near $-\infty$, $\mu^-(u, x) \sim y^{-1}$, $\mu^+(u, x) \sim -y^{-1}$, and the two eigenvectors are approximately $(0, 1, y^{-1}, 0)$ and $(0, 1, -y^{-1}, 0)$. This motivates the change of coordinates

$$u = u, \quad v = \tilde{v} + \tilde{w}, \quad w = y^{-1}\tilde{v} - y^{-1}\tilde{w}, \quad x = -y^{-2}. \tag{A.1}$$

The inverse of this coordinate change is

$$u = u, \quad \tilde{v} = \frac{1}{2}v - \frac{1}{2}(-x)^{-\frac{1}{2}}w, \quad \tilde{w} = \frac{1}{2}v + \frac{1}{2}(-x)^{-\frac{1}{2}}w, \quad y = -(-x)^{-\frac{1}{2}}. \tag{A.2}$$

In the new coordinates, (2.17)–(2.20) becomes

$$u_\xi = \tilde{v} + \tilde{w}, \tag{A.3}$$

$$\tilde{v}_\xi = \left(y^{-1} + \frac{3u^2y}{2} - \frac{\alpha}{2} - \frac{\epsilon y^2}{4}\right)\tilde{v} + \left(\frac{3u^2y}{2} + \frac{\alpha}{2} + \frac{\epsilon y^2}{4}\right)\tilde{w}, \tag{A.4}$$

$$\tilde{w}_\xi = \left(-\frac{3u^2y}{2} + \frac{\alpha}{2} + \frac{\epsilon y^2}{4}\right)\tilde{v} + \left(-y^{-1} - \frac{3u^2y}{2} - \frac{\alpha}{2} - \frac{\epsilon y^2}{4}\right)\tilde{w}, \tag{A.5}$$

$$y_\xi = -\frac{\epsilon y^3}{2}. \tag{A.6}$$

Since the terms y^{-1} in (A.3)–(A.6) are undefined at $y = 0$, we multiply (A.3)–(A.6) by $-y$, which is positive for $-\infty < y < 0$. We obtain

$$u_\xi = -y(\tilde{v} + \tilde{w}), \tag{A.7}$$

$$\tilde{v}_\xi = \left(-1 - \frac{y}{4}(6u^2 y - 2\alpha - \epsilon y^2)\right)\tilde{v} - \frac{y}{4}(6u^2 y + 2\alpha + \epsilon y^2)\tilde{w}, \tag{A.8}$$

$$\tilde{w}_\xi = \frac{y}{4}(6u^2 y - 2\alpha - \epsilon y^2)\tilde{v} + \left(1 + \frac{y}{4}(6u^2 + 2\alpha + \epsilon y^2)\right)\tilde{w}, \tag{A.9}$$

$$y_\xi = \frac{\epsilon y^4}{2}. \tag{A.10}$$

Consider the subset of $\mathcal{P}$ given by $\mathcal{P}_1 = \{(u, v, w, x) : |u| \le \frac{1}{\delta}, v = w = 0, -\infty < x \le -1\}$. It corresponds under the change of coordinates to $\{(u, \tilde{v}, \tilde{w}, y) : |u| \le \frac{1}{\delta}, \tilde{v} = \tilde{w} = 0, -1 \le y < 0\}$. The closure of the latter set is $\tilde{\mathcal{P}}_1 = \{(u, \tilde{v}, \tilde{w}, y) : |u| \le \frac{1}{\delta}, \tilde{v} = \tilde{w} = 0, -1 \le y \le 0\}$.

For $\epsilon = 0$, $\tilde{\mathcal{P}}_1$ is a compact set of equilibria of (A.7)–(A.10). Set $\epsilon = 0$ and linearize (A.7)–(A.10) at an equilibrium $(u, 0, 0, y)$. The matrix is

$$\begin{pmatrix} 0 & -y & -y & 0 \\ 0 & -1 - \frac{y}{4}(6u^2 y - 2\alpha) & -\frac{y}{4}(6u^2 y + 2\alpha) & 0 \\ 0 & \frac{y}{4}(6u^2 y - 2\alpha) & 1 + \frac{y}{4}(6u^2 + 2\alpha) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If $y = 0$ the eigenvalues are 0, 0, -1, 1. In fact, for $\epsilon = 0$, $\tilde{\mathcal{P}}_1$ is a compact normally hyperbolic invariant manifold of equilibria for (A.7)–(A.10). Therefore $\tilde{\mathcal{P}}_1$ remains a normally hyperbolic invariant manifold of (A.7)–(A.10) for a small $\epsilon > 0$. Its stable and unstable manifolds and their invariant foliations persist for small $\epsilon > 0$. These correspond to the the stable and unstable manifolds of $\mathcal{P}_1$ and their invariant foliations, which therefore also persist for small $\epsilon > 0$. It follows easily that the same is true for $\mathcal{P}$.

A.2. **The manifolds $\mathcal{K}$ and $\mathcal{M}_\epsilon$.** To treat $\mathcal{K}$ we consider the system (4.9)–(4.15). We make the coordinate changes (A.1) together with

$$U = U, \quad V = \tilde{V} + \tilde{W}, \quad W = y^{-1}\tilde{V} - y^{-1}\tilde{W}.$$

The rest of the discussion is similar to that for $\mathcal{P}$. The treatment of $\mathcal{M}_\epsilon$ is similar.

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695 USA, 919-515-6533

E-mail address: schecter@math.ncsu.edu

Department of Mathematics, Ferrum College, Box 1000, Ferrum, VA 24088 USA, 540-365-4398

E-mail address: mrtaylor@ferrum.edu