

The Mathematics of Lecture Hall Partitions

Carla D. Savage
Department of Computer Science
North Carolina State University
Raleigh, NC 27695-8206, USA
savage@ncsu.edu

May 1, 2016

Abstract

Over the past twenty years, lecture hall partitions have emerged as fundamental combinatorial structures, leading to new generalizations and interpretations of classical theorems and new results. In recent years, geometric approaches to lecture hall partitions have used polyhedral geometry to discover further properties of these rich combinatorial objects.

In this paper we give an overview of some of the surprising connections that have surfaced in the process of trying to understand the lecture hall partitions.

Keywords: theory of partitions; Eulerian polynomials; Ehrhart theory; lattice point enumeration; real-rooted polynomials; Gorenstein cones

Contents

1	The Lecture Hall Theorem	3
2	s-lecture hall partitions	4
3	Anti-lecture hall compositions	5
4	Truncated lecture hall partitions	6
5	The (k, ℓ)-lecture hall theorem	7
6	A generalization of Euler's partition theorem	10

7	(1,4)- and (4,1)- sequences and the little Göllnitz identities	12
8	The refined lecture hall theorem	14
9	An analog of the Gaussian polynomials for lecture hall partitions	15
10	Lecture hall partitions and permutations	16
11	Quadratic permutation statistics	17
12	Lattice point generating functions of lecture hall cones	19
13	Lecture hall polytopes and Ehrhart theory	20
14	Generalized lecture hall partitions and s-inversion sequences	21
15	s-Eulerian polynomials	24
16	Real-rooted polynomials	26
17	Generating lecture hall partitions by height	28
18	Fundamental parallelepipeds	30
19	Gorenstein cones	31
20	Concluding remarks	33

1 The Lecture Hall Theorem

A *partition* $p = (p_1, p_2, \dots, p_k)$ of an integer N is an unordered collection of positive integers p_i , called *parts*, such that $|p| = p_1 + p_2 + \dots + p_k = N$. Since the order of the parts does not matter, it is convenient to list the parts in nonincreasing order $p_1 \geq p_2 \geq \dots \geq p_k$. The number of partitions of N is the coefficient of q^N in $1/(q; q)_\infty$, where we use the notation $(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$, $(a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$, and $(a_1, \dots, a_k; q)_\infty = \prod_{j=1}^k (a_j; q)_\infty$.

Perhaps the best known theorem involving partitions is the following.

Theorem 1.1 (Euler's partition theorem). *The number of partitions of N in which all parts are distinct is equal to the number of partitions of N in which all parts are odd.*

The analytic form of Theorem 1.1 is

$$\prod_{i \geq 1} (1 + q^i) = \prod_{i \geq 1} \frac{1}{1 - q^{2i-1}}, \quad (1)$$

where the left-hand side is the sum of $q^{|p|}$ over partitions p into distinct parts and the right-hand side is the sum of $q^{|p|}$ over partitions into odd parts.

The *lecture hall partitions* were introduced by Bousquet-Mélou and Eriksson in [8] and defined by:

$$\mathbf{L}_n = \left\{ \lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \dots \leq \frac{\lambda_n}{n} \right\}. \quad (2)$$

The conditions on λ_i can be viewed as height constraints on the i th row of a lecture hall in order that each student can see over the head of the students in previous rows to view the shoes of her teacher.

In 1997, Bousquet-Mélou and Eriksson proved the following.

Theorem 1.2 (The lecture hall theorem [8]). *The number of lecture hall partitions of N in \mathbf{L}_n is equal to the number of partitions of N into odd parts less than $2n$.*

The generating function version of Theorem 1.2 is:

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lambda|} = \prod_{i=1}^n \frac{1}{1 - q^{2i-1}}. \quad (3)$$

If we reverse a lecture hall partition and ignore the parts of size 0, we do actually get a *partition* into distinct parts. In this sense, as $n \rightarrow \infty$, \mathbf{L}_n becomes the set of partitions into distinct parts. Theorem 1.2, therefore, is a finite version of Theorem 1.1, and its discovery came as an unexpected surprise when Bousquet-Mélou and Eriksson introduced it in 1997.

Whereas Theorem 1.1 is well-understood analytically, combinatorially, and algebraically, Theorem 1.2 is hardly understood at all. This is in spite of the fact that by now there are many proofs, including those of Bousquet-Mélou and Eriksson [8, 9, 10], Andrews [1], Yee [55, 56], Andrews, Paule, Riese, and Strehl [3], Eriksen [31], and Bradford et al [11]. We have also contributed to the collection of proofs with co-authors Corteel [25], Corteel and Lee [20], Andrews and Corteel [2], Bright [15], and, most recently, Corteel and Lovejoy [23].

In the search for a transparent proof for the lecture hall theorem, many connections have been discovered. Over the past twenty years, lecture hall partitions have emerged as fundamental structures in combinatorics, number theory, algebra, and geometry, leading to new generalizations and interpretations of classical theorems and new results.

In this paper we give an overview of some of the surprising connections that have surfaced and results that have been discovered in the process of trying to understand lecture hall partitions.

2 \mathbf{s} -lecture hall partitions

For any sequence \mathbf{s} of positive integers, define the \mathbf{s} -lecture hall partitions by

$$\mathbf{L}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \right\}. \quad (4)$$

The name is a slight abuse of notation, since unless \mathbf{s} is nondecreasing, $\lambda \in \mathbf{L}_n^{(\mathbf{s})}$ may not be monotone and therefore may be a composition of $|\lambda|$ rather than a partition. For example, $(2, 3)$ and $(3, 2)$ are distinct “ $(5, 3)$ -lecture hall partitions” of 5 since $0 \leq 2/5 \leq 3/3$ and $0 \leq 3/5 \leq 2/3$.

The \mathbf{s} -lecture hall partitions were first considered by Bousquet-Mélou and Eriksson in [9], where \mathbf{s} was required to be nondecreasing. They called a sequence \mathbf{s} *polynomic* if

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} = \prod_{i=1}^n \frac{1}{1 - q^{d_i}} \quad (5)$$

for some positive integers d_1, \dots, d_n .

In [9] Bousquet-Mélou and Eriksson discovered an infinite family of polynomic sequences - the so-called (k, ℓ) -sequences - which are discussed in Section 5.

In Sections 3 and 4 we show that there are some sequences \mathbf{s} which are not polynomic, but nevertheless give rise to interesting generating functions.

3 Anti-lecture hall compositions

In the search for sequences \mathbf{s} for which the \mathbf{s} -lecture hall partitions have an interesting generating function, it is natural to consider $\mathbf{s} = (n, n-1, \dots, 1)$.

Define the *anti-lecture hall compositions* \mathbf{A}_n by

$$\mathbf{A}_n = \left\{ \lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{n} \leq \frac{\lambda_2}{n-1} \leq \dots \leq \frac{\lambda_n}{1} \right\}. \quad (6)$$

So $\mathbf{A}_n = \mathbf{L}_n^{(n, n-1, \dots, 1)}$. For example $(1, 2, 3, 4)$, $(4, 3, 2, 1)$, and $(1, 4, 3, 2)$ are all in \mathbf{A}_4 , but $(5, 3, 2, 1)$ is not. The generating function for \mathbf{A}_n has a simple product form.

Theorem 3.1 (The anti-lecture hall theorem (Corteel, S [24])).

$$A_n(q) := \sum_{\lambda \in \mathbf{A}_n} q^{|\lambda|} = \prod_{i=1}^n \frac{1+q^i}{1-q^{i+1}} = \frac{(-q; q)_n}{(q^2; q)_n}. \quad (7)$$

The product in (7) suggests a connection with the overpartitions [22] of Corteel and Lovejoy. An *overpartition* of N is a nonincreasing sequence of positive numbers whose sum is N in which the first occurrence of a number may be overlined. Clearly, summing over all overpartitions λ ,

$$\sum_{\lambda} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1+q^i}{1-q^i} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}. \quad (8)$$

Comparing (8) with (7), letting $n \rightarrow \infty$ in (7), and ignoring the ‘0’ parts, we have the following.

Observation 3.2. *The number of anti-lecture hall compositions of N (of any length) is equal to the number of overpartitions of N with no un-overlined 1.*

The first hint that there might be a deeper connection between anti-lecture hall compositions and overpartitions is this surprising result of Chen, Sang and Shi [16]. Let \mathcal{A}_t be the set of anti-lecture hall compositions (of any length k) into positive parts with last part at most t :

$$\mathcal{A}_t : 0 < \frac{\lambda_1}{k} \leq \frac{\lambda_2}{k-1} \leq \dots \leq \frac{\lambda_k}{1} \leq t.$$

Theorem 3.3 (Chen, Sang, Shi [16]). *The number of anti-lecture hall compositions of N in \mathcal{A}_t is the same as the number of overpartitions of N with no un-overlined parts congruent to $0, \pm 1$ modulo $t+2$. That is,*

$$\sum_{\lambda \in \mathcal{A}_t} q^{|\lambda|} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} (q, q^{t+1}, q^{t+2}; q^{t+2})_{\infty}. \quad (9)$$

An alternative approach and a generalization of Theorem 3.3, is given with Corteel and Lovejoy in [23]. The method of [23] derives a combinatorial recurrence for a (finite) version of a 3-parameter refinement and solves it using Andrews' generalization of the Watson-Whipple transformation.

In [30], Engström and Stamps make use of anti-lecture hall compositions to give a geometric picture of Betti diagrams of ideals with 2-linear resolutions and thereby to prove that any Betti diagram of a module with a 2-linear resolution arises from a direct sum of Stanley Reisner rings constructed from threshold graphs.

4 Truncated lecture hall partitions

Computations suggested that limiting the number of positive parts of $\lambda \in \mathbf{L}_n$ would give rise to an interesting refinement of Theorem 1.2. Let $\mathbf{L}_{n,k}$ be the set of $\lambda \in \mathbf{L}_n$ with at most k positive parts. The elements of $\mathbf{L}_{n,k}$ are *truncated lecture hall partitions*. Note that $\mathbf{L}_{n,k} = \mathbf{L}_k^{(n-k+1, n-k+2, \dots, n)}$.

Theorem 4.1 (The truncated lecture hall theorem (Corteel, S [25])). *The number of lecture hall partitions of N in $\mathbf{L}_{n,k}$ is equal to the number of partitions of N into odd parts less than $2n$, with the following constraint on the parts: at most $\lfloor k/2 \rfloor$ of the odd parts can be chosen from the interval $[2 \lfloor k/2 \rfloor + 1, 2(n - \lfloor k/2 \rfloor) - 1]$.*

The proof in [25] uses q -calculus. A combinatorial proof of Theorem 4.1 appears in Appendix 1 of [47].

For a nonnegative integer n , let $[n]_q = \frac{1-q^n}{1-q}$ and $[n]_q! = \prod_{i=1}^n [i]_q$. The q -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Let $\mathbf{A}_{n,k}$ be the set of *truncated anti-lecture hall compositions* defined by $\mathbf{A}_{n,k} = \mathbf{L}_k^{(n, n-1, \dots, n-k+1)}$. Let $\bar{\mathbf{L}}_{n,k}$ be the set of $\lambda \in \mathbf{L}_{n,k}$ with *exactly* k positive parts. The generating functions for $\bar{\mathbf{L}}_{n,k}$ and $\mathbf{A}_{n,k}$ have the following product forms.

Theorem 4.2 (Corteel, S [25]).

$$\sum_{\lambda \in \bar{\mathbf{L}}_{n,k}} q^{|\lambda|} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1}; q)_k}{(q^{2n-k+1}; q)_k}; \quad (10)$$

$$\sum_{\lambda \in \mathbf{A}_{n,k}} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-q^{n-k+1}; q)_k}{q^{2(n-k+1)} (q; q)_k}. \quad (11)$$

In many cases, enumerating lecture hall partitions can be accomplished using combinatorial reasoning to devise a recurrence for a multi-parameter refinement, followed by application of the right q -series identity to solve the recurrence. Andrews' generalization of the Watson-Whipple transformation was mentioned in this context in Section 3. This was also the case for the identities in Theorem 4.2 which were proven in [25] with the help of the q -Chu Vandermonde identities:

$$\frac{a^n(c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \frac{(a; q)_m (q^{-n}; q)_m}{(c; q)_m (q; q)_m} q^m$$

$$\frac{(c/a; q)_n}{(c; q)_n} = \sum_{m=0}^n \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(a; q)_m}{(c; q)_m} (-c/a)^m q^{\binom{m}{2}}.$$

A proof of Theorem 4.2 appearing in [20] derives different recurrences, but solves them with the help of the same q -Chu Vandermonde identities. Yet another pair of recurrences were derived in [21] and solved in [2] using two new finite corollaries of a q -analog of Gauss's second theorem.

5 The (k, ℓ) -lecture hall theorem

Define the (k, ℓ) sequence $a^{(k, \ell)}$ by

$$a_{2i}^{(k, \ell)} = \ell a_{2i-1}^{(k, \ell)} - a_{2i-2}^{(k, \ell)} \tag{12}$$

$$a_{2i+1}^{(k, \ell)} = k a_{2i}^{(k, \ell)} - a_{2i-1}^{(k, \ell)}, \tag{13}$$

with initial conditions $a_1^{(k, \ell)} = 1$, $a_2^{(k, \ell)} = \ell$. When $k = \ell = 2$, $a^{(k, \ell)}$ is the sequence of positive integers.

In [9] Bousquet-Mélou and Eriksson show that the generating function for the $a^{(k, \ell)}$ -lecture hall partitions is a natural generalization of (3). We'll describe the result, sketch the proof, and show how it gives rise to a recursive bijection. In the special case $k = \ell = 2$, this is a proof (our favorite) of the lecture hall theorem.

For this section it will be convenient to reverse the labeling of the parts of an $a^{(k, \ell)}$ -lecture hall partition. To avoid confusion, define

$$\mathbf{G}_n^{(k, \ell)} = \left\{ \lambda \in \mathbb{Z}^n \mid \frac{\lambda_1}{a_n^{(k, \ell)}} \geq \frac{\lambda_2}{a_{n-1}^{(k, \ell)}} \geq \dots \geq \frac{\lambda_n}{a_1^{(k, \ell)}} \geq 0 \right\}$$

For $\lambda \in \mathbf{G}_n^{(k, \ell)}$ define $|\lambda|_o = \lambda_1 + \lambda_3 + \dots$ and $|\lambda|_e = \lambda_2 + \lambda_4 + \dots$. Define

$$G_n^{(k, \ell)}(x, y) = \sum_{\lambda \in \mathbf{G}_n^{(k, \ell)}} x^{|\lambda|_o} y^{|\lambda|_e}.$$

Bousquet-Mélou and Eriksson proved the following theorem for $k \geq 2$ and $\ell \geq 2$. It is shown in [26] that the result holds as well for $(k, \ell) \in \{(1, 4), (4, 1)\}$.

Theorem 5.1 (The (k, ℓ) -lecture hall theorem (Bousquet-Mélou, Eriksson [9])). *For positive integers k, ℓ with $k\ell \geq 4$,*

$$G_{2n}^{(k, \ell)}(x, y) = \prod_{i=1}^{2n} \frac{1}{1 - x^{a_i^{(k, \ell)}} y^{a_{i-1}^{(\ell, k)}}}$$

$$G_{2n-1}^{(k, \ell)}(x, y) = \prod_{i=1}^{2n-1} \frac{1}{1 - x^{a_i^{(\ell, k)}} y^{a_{i-1}^{(k, \ell)}}}.$$

Proof. We outline the clever approach of Bousquet-Mélou and Eriksson. The description below is adapted from [26], Appendix 1.

The strategy is to show that the following recurrence holds:

$$G_n^{(k, \ell)}(x, y) = G_{n-1}^{(k, \ell)}(x^\ell y, x^{-1}) / (1 - x) \quad \text{if } n \text{ is even} \quad (14)$$

$$G_n^{(k, \ell)}(x, y) = G_{n-1}^{(k, \ell)}(x^k y, x^{-1}) / (1 - x) \quad \text{if } n \text{ is odd,} \quad (15)$$

with initial condition $G_0^{(k, \ell)}(x, y) = 1$. It can then be shown by induction, using the properties of $a^{(k, \ell)}$, that the theorem follows.

Recurrences (14) and (15) are derived as follows. To simplify notation, let $a_n = a_n^{(k, \ell)}$. Define

$$\Gamma_n : \mathbf{G}_{n-1}^{(k, \ell)} \times \mathbb{N} \rightarrow \mathbf{G}_n^{(k, \ell)}. \quad (16)$$

Given $(\lambda, s) \in \mathbf{G}_{n-1}^{(k, \ell)} \times \mathbb{N}$, let $\Gamma_n(\lambda, s) = \mu$, where

$$\begin{aligned} \mu_1 &= \left\lfloor \frac{a_n \lambda_1}{a_{n-1}} \right\rfloor + s \\ \mu_{2t} &= \lambda_{2t-1}, \quad 1 \leq t \leq n/2; \\ \mu_{2t+1} &= \begin{cases} \left\lfloor \frac{a_{n-2t} \lambda_{2t+1}}{a_{n-2t-1}} \right\rfloor + \left\lfloor \frac{a_{n-2t} \lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t}, & 1 \leq t < (n-1)/2 \\ \left\lfloor \frac{a_{n-2t} \lambda_{2t-1}}{a_{n-2t+1}} \right\rfloor - \lambda_{2t} & t = (n-1)/2. \end{cases} \end{aligned}$$

One then uses the properties of $a^{(k, \ell)}$ to prove that $\mu \in \mathbf{G}_n^{(k, \ell)}$, that Γ_n is a bijection, and that

$$\begin{aligned} |\mu|_e &= |\lambda|_o; \\ |\mu|_o &= \begin{cases} \ell |\lambda|_o - |\lambda|_e + s & \text{if } n \text{ is even} \\ k |\lambda|_o - |\lambda|_e + s & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Then, when n is even,

$$\begin{aligned}
G_n^{(k,\ell)}(x,y) &= \sum_{\mu \in \mathbf{G}_n^{(k,\ell)}} x^{|\mu|_o} y^{|\mu|_e} = \sum_{\lambda \in \mathbf{G}_{n-1}^{(k,\ell)}} \sum_{s=0}^{\infty} x^{\ell|\lambda|_o - |\lambda|_e + s} y^{|\lambda|_o} \\
&= \frac{1}{1-x} \sum_{\lambda \in \mathbf{G}_{n-1}^{(k,\ell)}} (x^\ell y)^{|\lambda|_o} ((1/x)^{|\lambda|_e}) \\
&= \frac{G_{n-1}^{(k,\ell)}(x^\ell y, x^{-1})}{1-x},
\end{aligned}$$

giving (14). When n is odd, similar reasoning gives (15). \square

Fix k, ℓ and let $\rho_i = a_i^{(k,\ell)} + a_{i-1}^{(\ell,k)}$ and $r_i = a_i^{(\ell,k)} + a_{i-1}^{(k,\ell)}$, with $\rho_1 = r_1 = 1$. As noted in [8, 9] the proof of Theorem 5.1 gives rise to a recursive weight-preserving bijection that we refer to here as BME:

$$\text{BME}_n : \mathbf{G}_n^{(k,\ell)} \rightarrow \begin{cases} \text{partitions into parts from } \{\rho_1, \dots, \rho_n\} & \text{if } n \text{ is even} \\ \text{partitions into parts from } \{r_1, \dots, r_n\} & \text{if } n \text{ is odd.} \end{cases} \quad (17)$$

Represent a partition α into parts from $\{\rho_1, \dots, \rho_n\}$ as $\alpha = \rho_n^{m_n} \rho_{n-1}^{m_{n-1}} \dots \rho_1^{m_1}$ where m_i is the number of copies of ρ_i in α . Do likewise when the parts are chosen instead from $\{r_1, \dots, r_n\}$.

For $\mu \in \mathbf{G}_n^{(k,\ell)}$, define $\text{BME}_n(\mu)$ as follows, using Γ_n from (16).

If $n = 0$, μ is the empty partition and so is $\text{BME}_n(\mu)$. Otherwise,

if $\mu = \Gamma_n(\lambda, s)$ then

$$\begin{aligned}
n \text{ even:} & \text{ if } \text{BME}_{n-1}(\lambda) = r_{n-1}^{m_{n-1}} r_{n-2}^{m_{n-2}} \dots r_1^{m_1} \text{ then } \text{BME}_n(\mu) = \rho_n^{m_{n-1}} \rho_{n-1}^{m_{n-2}} \dots \rho_2^{m_1} \rho_1^s; \\
n \text{ odd:} & \text{ if } \text{BME}_{n-1}(\lambda) = \rho_{n-1}^{m_{n-1}} \rho_{n-2}^{m_{n-2}} \dots \rho_1^{m_1} \text{ then } \text{BME}_n(\mu) = r_n^{m_{n-1}} r_{n-1}^{m_{n-2}} \dots r_2^{m_1} r_1^s.
\end{aligned}$$

BME is simple to implement and useful for experiments. As an example:

$$\begin{aligned}
\Gamma_2((0), 4) &= (4, 0) \in \mathbf{G}_2^{(1,4)} \\
\Gamma_3((4, 0), 1) &= (4, 4, 1) \in \mathbf{G}_3^{(1,4)} \\
\Gamma_4((4, 4, 1), 1) &= (12, 4, 5, 1) \in \mathbf{G}_4^{(1,4)} \\
\Gamma_5((12, 4, 5, 1), 1) &= (9, 12, 4, 5, 0) \in \mathbf{G}_5^{(1,4)}
\end{aligned}$$

and so, since $(\rho_1, \dots, \rho_5) = (1, 5, 4, 11, 7)$ and $(r_1, \dots, r_5) = (1, 2, 7, 5, 13)$,

$$\begin{aligned} \text{BME}_1(0) &= 1^0 \\ \text{BME}_2(4, 0) &= 5^0 1^4 \\ \text{BME}_3(4, 4, 1) &= 7^0 2^4 1^1 \\ \text{BME}_4(12, 4, 5, 1) &= 11^0 4^4 5^1 1^1 \\ \text{BME}_5(9, 12, 4, 5, 0) &= 13^0 5^4 7^1 2^1 1^1. \end{aligned}$$

6 A generalization of Euler's partition theorem

Define the ℓ -sequence $a^{(\ell)}$ to be the (k, ℓ) -sequence of (12), (13) with $k = \ell$. Then

$$a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)},$$

with $a_1^{(\ell)} = 1$, $a_2^{(\ell)} = \ell$. In this case, Theorem 5.1 takes the following form.

Theorem 6.1 (The ℓ -lecture hall theorem (Bousquet-Mélou, Eriksson [9])). *For $\ell \geq 2$ and $n \geq 0$,*

$$\sum_{\lambda} q^{\lambda_1 + \lambda_2 + \dots + \lambda_n} = \prod_{i=1}^n \frac{1}{1 - q^{a_i^{(\ell)} + a_{i-1}^{(\ell)}}}$$

where the sum is over all integer sequences $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfying

$$\frac{\lambda_1}{a_n^{(\ell)}} \geq \frac{\lambda_2}{a_{n-1}^{(\ell)}} \geq \dots \geq \frac{\lambda_n}{a_1^{(\ell)}} \geq 0.$$

Let c_ℓ be the largest root of the characteristic polynomial $x^2 - \ell x + 1$. Since $\lim_{n \rightarrow \infty} a_n^{(\ell)} / a_{n-1}^{(\ell)} = c_\ell$, observe that as $n \rightarrow \infty$, Theorem 6.1 becomes the following generalization of Theorem 1.1.

Theorem 6.2 (The ℓ -Euler theorem (Bousquet-Mélou, Eriksson [9])). *The number of partitions of an integer N into parts from the set*

$$\{a_i^{(\ell)} + a_{i-1}^{(\ell)} \mid i \geq 1\}$$

is the same as the number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of N in which $\lambda_i / \lambda_{i+1} > c_\ell$ for all consecutive positive parts λ_i, λ_{i+1} .

When $\ell = 2$, $a^{(\ell)}$ is the sequence of positive integers, $a_i^{(\ell)} + a_{i-1}^{(\ell)} = 2i - 1$ and $c_\ell = 1$. So Theorem 6.1 becomes the lecture hall theorem and Theorem 6.2 becomes Euler's partition theorem.

In [47], with Yee, we provide a simple bijection $\Theta^{(\ell)}$ to prove Theorem 6.2. The bijection $\Theta^{(\ell)}$ has several nice properties:

- $\Theta^{(2)}$ is Sylvester's bijection for Euler's partition theorem (Theorem 1.1).
- A finite version $\Theta_n^{(\ell)}$ of $\Theta^{(\ell)}$ proves the ℓ -lecture hall theorem (Theorem 6.1).
- $\Theta_n^{(2)}$ is the same bijection for Theorem 1.2 as the one due to Yee in [55].
- $\Theta_n^{(2)}$ allows for a combinatorial proof of Theorem 4.1, the truncated lecture hall theorem.
- It appears that $\Theta_n^{(\ell)} = \text{BME}_n$ (17) for $k = \ell$, but this has not been proven.

We describe $\Theta^{(\ell)}$, to show how simple it is. Fix $\ell \geq 2$ and let $a_i = a_i^{(\ell)}$. For positive integer i , let $p_i = a_i + a_{i-1}$. Let $O^{(\ell)}$ be the set of partitions (of any integer) into parts from the set $\{p_1, p_2, \dots\}$. Represent a partition $\mu \in O^{(\ell)}$ as $\mu = p_r^{m_r} p_{r-1}^{m_{r-1}} \cdots p_1^{m_1}$ where m_i is the number of copies of p_i in μ .

Let $D^{(\ell)}$ be the set of partitions (of any integer) into a decreasing sequence of positive parts with the property the ratio of consecutive positive parts is at least c_ℓ .

Bijection for the ℓ -Euler theorem (S, Yee, [47]) $\Theta^{(\ell)} : O^{(\ell)} \rightarrow D^{(\ell)}$

For $\mu = p_r^{m_r} p_{r-1}^{m_{r-1}} \cdots p_1^{m_1} \in O^{(\ell)}$, define $\Theta^{(\ell)}(\mu)$ to be the sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ obtained from the empty partition $(0, 0, \dots)$ by inserting the parts of μ in nonincreasing order, one at a time, largest to smallest, according to the following $\Theta^{(\ell)}$ insertion procedure.

To insert p_k into $(\lambda_1, \lambda_2, \dots)$:

If $k = 1$ then $\lambda_1 \leftarrow \lambda_1 + a_1$;

otherwise, if $\lambda_1 + a_k - a_{k-1} > c_\ell(\lambda_2 + a_{k-1} - a_{k-2})$ then

$\lambda_1 \leftarrow \lambda_1 + a_k - a_{k-1}$;

$\lambda_2 \leftarrow \lambda_2 + a_{k-1} - a_{k-2}$;

recursively insert p_{k-1} into $(\lambda_3, \lambda_4, \dots)$ via $\Theta^{(\ell)}$ insertion;

otherwise, $\lambda_1 \leftarrow \lambda_1 + a_k$; $\lambda_2 \leftarrow \lambda_2 + a_{k-1}$;

The most mysterious thing involved in proving that $\Theta^{(\ell)}$ is a bijection is this: If we follow this insertion procedure how do we know that $\lambda_2 > c_\ell \lambda_3$? It is *true*, but surprisingly difficult to prove. The proof method involves encoding integers i as words $w(i)$ with the property that $i > c_\ell j$ if and only if $w(i) > w(j)$ in lexicographic order.

Question: Is there an easier proof that $\Theta^{(\ell)}$ is a bijection?

The bijection $\Theta_n^{(\ell)} : O_n^{(\ell)} \rightarrow D_n^{(\ell)}$ for the ℓ -lecture hall theorem, Theorem 6.1, involves a slight modification of $\Theta^{(\ell)}$, and can be found in [47].

7 (1, 4)- and (4, 1)- sequences and the little Göllnitz identities

Consider the following (k, ℓ) -sequences defined in Section 5:

$$a^{(1,4)} = 1, 4, 3, 8, 5, 12, 7, 16, 9, \dots; \quad (18)$$

$$a^{(4,1)} = 1, 1, 3, 2, 5, 3, 7, 4, 9, \dots \quad (19)$$

The sequences satisfy

$$a_{2i+1}^{(1,4)} = 2i + 1; \quad a_{2i}^{(1,4)} = 4i \quad (20)$$

$$a_{2i+1}^{(4,1)} = 2i + 1; \quad a_{2i}^{(4,1)} = i \quad (21)$$

and the associated lecture hall sequences satisfy

$$\mathbf{G}_{2k}^{(1,4)} : \frac{\lambda_1}{4k} \geq \frac{\lambda_2}{2k-1} \geq \frac{\lambda_3}{4(k-1)} \geq \frac{\lambda_4}{2k-3} \geq \dots \geq \frac{\lambda_{2k}}{1} \geq 0;$$

$$\mathbf{G}_{2k}^{(4,1)} : \frac{\lambda_1}{k} \geq \frac{\lambda_2}{2k-1} \geq \frac{\lambda_3}{k-1} \geq \frac{\lambda_4}{2k-3} \geq \dots \geq \frac{\lambda_{2k}}{1} \geq 0.$$

By Theorem 5.1,

$$G_{2k}^{(1,4)}(x, y) = \prod_{i=1}^k \frac{1}{(1 - x^{2i-1}y^{i-1})(1 - x^{4i}y^{2i-1})} \quad (22)$$

and

$$G_{2k}^{(4,1)}(x, y) = \prod_{i=1}^k \frac{1}{(1 - x^{2i-1}y^{4i-4})(1 - x^i y^{2i-1})}. \quad (23)$$

Note that in $\mathbf{G}_{2k}^{(1,4)}$, the ratio of consecutive positive terms λ_i/λ_{i+1} for i odd is at least $(4j)/(2j-1) > 2$ for some j or, if i is even, at least $(2j-1)/(4(j-1)) > 1/2$ for some j . Conversely, every finite sequence λ of positive integers with $\lambda_i/\lambda_{i+1} > 2$ for odd i and $\lambda_i/\lambda_{i+1} > 1/2$ for even i will be in $\mathbf{G}_{2k}^{(1,4)}$ for k large enough (appending zeroes to λ , if necessary). Thus $\lim_{k \rightarrow \infty} \mathbf{G}_{2k}^{(1,4)}$ is the set of sequences (of any even length) satisfying

$$\frac{\lambda_1}{2} > \frac{\lambda_2}{1} > \frac{\lambda_3}{2} > \frac{\lambda_4}{1} > \frac{\lambda_5}{2} > \frac{\lambda_6}{1} > \dots \quad (24)$$

so the generating function for the sequences λ satisfying (24) is given by taking the limit in (22):

$$\lim_{k \rightarrow \infty} G_{2k}^{(1,4)}(x, y) = \frac{1}{(x; x^2y)_{\infty}(x^4y; x^4y^2)_{\infty}}. \quad (25)$$

Similarly, $\lim_{k \rightarrow \infty} \mathbf{G}_{2k}^{(4,1)}$ is the set of sequences (of any even length) satisfying

$$\frac{\lambda_1}{1} > \frac{\lambda_2}{2} > \frac{\lambda_3}{1} > \frac{\lambda_4}{2} > \frac{\lambda_5}{1} > \frac{\lambda_6}{2} > \dots \quad (26)$$

whose generating function is found by taking the limit in (23):

$$\lim_{k \rightarrow \infty} G_{2k}^{(4,1)}(x, y) = \frac{1}{(x; x^2 y^4)_\infty (xy; xy^2)_\infty}. \quad (27)$$

In [26], with Corteel and Sills, we used combinatorial methods to directly compute sum-form generating functions for sequences satisfying (24) and (26). These generating functions must be equal to those in (25) and (27) respectively, with $x = q$, $y = q$. In this way we discovered the new pair of identities whose connection to lecture hall partitions is hidden.

Theorem 7.1 (Corteel, S, Sills [26]).

$$\sum_{j=0}^{\infty} q^{j(3j-1)/2} \frac{(q^2; q^6)_j}{(q; q)_{3j}} = \frac{1}{(q; q^3)_\infty (q^5; q^6)_\infty} \quad (28)$$

$$\sum_{j=0}^{\infty} q^{j(3j+1)/2} \frac{(q^4; q^6)_j}{(q; q)_{3j+1}} = \frac{1}{(q^2; q^3)_\infty (q; q^6)_\infty} \quad (29)$$

Another consequence of (22) and (23) is the following, which also does not mention lecture hall partitions.

Theorem 7.2 (S, Sills [44]). *The number of partitions λ of an integer N into distinct parts $\lambda_1 > \lambda_2 > \dots$ such that all parts λ_{2i} are even is equal to the number of partitions of N into parts congruent to 1, 5, or 6 (mod 8).*

Proof. The map sending a sequence λ satisfying (24) to the partition $(\lambda_1, 2\lambda_2, \lambda_3, 2\lambda_4, \dots)$ is a bijection so replacing y by y^2 in (22) and then setting $x = y = q$ gives the result. \square

Similarly, from (23) we get the following.

Theorem 7.3 ([44]). *The number of partitions λ of an integer N into distinct parts $\lambda_1 > \lambda_2 > \dots$ such that all parts λ_{2i-1} are even is equal to the number of partitions of N into parts congruent to 2, 3, or 7 (mod 8).*

(See [26] and [44] for refinements and connections with Heine's q -Gauss summation.)

Compare Theorems 7.2 and 7.3 to the identities below, known as "Göllnitz's little partition theorems" [34].

Theorem 7.4 (Göllnitz [34]). *The number of partitions of N into parts differing by at least 2 and no consecutive odd parts equals the number of partitions of N into parts congruent to 1, 5, or 6 (mod 8).*

Theorem 7.5 (Göllnitz [34]). *The number of partitions of N into parts differing by at least 2, no consecutive odd parts, and no ones equals the number of partitions of N into parts congruent to 2, 3, or 7 (mod 8).*

Question: Is there a simple bijection between the partitions in Theorems 7.2 and 7.4? Between the partitions in Theorems 7.3 and 7.5?

Theorems 7.2 and 7.3 have been significantly extended and generalized in a recent paper by Berkovich and Uncu [7].

8 The refined lecture hall theorem

The first proof of the lecture hall theorem in the original paper of Bousquet-Mèlou and Eriksson was derived from Bott's formula for the Poincaré series of the affine Weyl group \tilde{C}_n . In their third paper on lecture hall partitions, Bousquet-Mèlou and Eriksson proved the following, which was suggested by a refinement of Bott's formula.

Theorem 8.1 (The refined lecture hall theorem) (Bousquet-Mèlou, Eriksson [10]).

$$L_n(q, u, v) = \sum_{\lambda \in \mathbf{L}_n} q^{|\lambda|} u^{|\lceil \lambda \rceil} v^{o(\lceil \lambda \rceil)} = \prod_{i=1}^n \frac{1 + uvq^i}{1 - u^2q^{n+i}} \quad (30)$$

where $\lceil \lambda \rceil = (\lceil \lambda_1/1 \rceil, \lceil \lambda_2/2 \rceil, \dots, \lceil \lambda_n/n \rceil)$ and $o(\lambda)$ is the number of odd parts of λ .

In Section 10, which establishes a connection between lecture hall partitions and permutations, it will be seen that $|\lceil \lambda \rceil|$ on lecture hall partitions corresponds to a Mahonian statistic on permutations and $\lceil \lambda_n/n \rceil$ corresponds to an Eulerian statistic.

An analog of Theorem 8.1 holds for anti-lecture hall compositions:

Theorem 8.2 (The refined anti-lecture hall theorem) (Corteel, S [24]).

$$A_n(q, u, v) = \sum_{\lambda \in \mathbf{A}_n} q^{|\lambda|} u^{|\lfloor \lambda \rfloor} v^{o(\lfloor \lambda \rfloor)} = \prod_{i=1}^n \frac{1 + uvq^i}{1 - u^2q^{i+1}} \quad (31)$$

where $\lfloor \lambda \rfloor = (\lfloor \lambda_1/1 \rfloor, \lfloor \lambda_2/2 \rfloor, \dots, \lfloor \lambda_n/n \rfloor)$.

The generating functions (10) and (11) for truncated lecture hall partitions and truncated anti-lecture hall compositions can be similarly refined, as shown in [25].

Theorem 8.3 (Corteel, S [25]).

$$\bar{L}_{n,k}(q, u, v) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{((-u/v)q^{n-k+1}; q)_k}{(u^2q^{2n-k+1}; q)_k}; \quad (32)$$

$$A_{n,k}(q, u, v) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1}; q)_k}{(u^2q^{2(n-k+1)}; q)_k}. \quad (33)$$

Question: Is there a refinement of Bott's formula corresponding to (32)?

Equations (30) and (31) (dropping the v) imply the relationship

$$L_n(q, u) = A_n(q^{-1}, uq^{n+1}). \quad (34)$$

The search for a combinatorial explanation for the relationship (34) motivated some of the work in the next few sections. A geometric view proved successful in [15].

9 An analog of the Gaussian polynomials for lecture hall partitions

The generating function for $P_{n \times k}$, the set of partitions with at most n parts and with largest part at most k , is given by the Gaussian polynomial:

$$\sum_{p \in P_{n \times k}} q^{|p|} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q. \quad (35)$$

We get an interesting result if we similarly consider the lecture hall partitions in \mathbf{L}_n with largest part at most k . Write k as $k = tn + i$ where $0 \leq i < n$.

Theorem 9.1 (Corteel, Lee, S [20]).

$$|\{\lambda \in \mathbf{L}_n \mid \lambda_n \leq tn + i\}| = (t+1)^{n-i}(t+2)^i. \quad (36)$$

For fixed n , this is a polynomial in k for fixed i . In particular, when $i = 0$ we have:

$$|\{\lambda \in \mathbf{L}_n \mid \frac{\lambda_n}{n} \leq t\}| = (t+1)^n. \quad (37)$$

A similar result holds for anti-lecture hall compositions [20]:

$$|\{\lambda \in \mathbf{A}_n \mid \frac{\lambda_n}{1} \leq t\}| = (t+1)^n \quad (38)$$

because of the following.

Theorem 9.2 (Corteel, Lee, S [20]). *For any sequence \mathbf{s} of positive integers,*

$$|\{\lambda \in \mathbf{L}_n^{(s_1, \dots, s_n)} \mid \frac{\lambda_n}{s_n} \leq t\}| = |\{\lambda \in \mathbf{L}_n^{(s_n, \dots, s_1)} \mid \frac{\lambda_n}{s_1} \leq t\}| \quad (39)$$

The fact that $(t+1)^n$ is the number of points in the unit cube suggests a connection with permutations. In establishing the relationship between lecture hall partitions and permutations in the next section, the following u -analog of (36), in which u tracks the statistic $|\lambda|$ from Section 8, will be relevant.

Theorem 9.3 (S, Schuster [43]).

$$\sum_{\lambda \in \{\lambda \in \mathbf{L}_n \mid \lambda_n \leq tn+i\}} u^{|\lambda|} = ([t+1]_u)^{n-i} ([t+2]_u)^i. \quad (40)$$

Only recently, with Corteel and Lovejoy, were we able to give (q, u, v) -refinements of equations (37) and (38) [23].

10 Lecture hall partitions and permutations

In this section we relate statistics on lecture hall partitions to Eulerian and Mahonian statistics on permutations.

The *Eulerian polynomial*, $E_n(x)$, defined by

$$\sum_{t \geq 0} (t+1)^n x^t = \frac{E_n(x)}{(1-x)^{n+1}}, \quad (41)$$

has the combinatorial interpretation

$$E_n(x) = \sum_{\pi \in \mathbf{S}_n} x^{\text{des } \pi} \quad (42)$$

where \mathbf{S}_n is the set of permutations of $1, \dots, n$ and for $\pi \in \mathbf{S}_n$, $\text{Des } \pi$ is the set of *descents* of π , i.e. the set of positions i for which $\pi_i > \pi_{i+1}$, and $\text{des } \pi = |\text{Des } \pi|$.

The *major index* of $\pi \in \mathbf{S}_n$ is the statistic $\text{maj } \pi = \sum_{i \in \text{Des } \pi} i$. The joint distribution of des and maj over permutations has the following form. This is a special case of a result of MacMahon.

Theorem 10.1 (MacMahon, [39], Vol. 2. p. 211).

$$\sum_{t \geq 0} ([t+1]_u)^n x^t = \frac{\sum_{\pi \in \mathbf{S}_n} x^{\text{des } \pi} u^{\text{maj } \pi}}{\prod_{i=0}^n (1-xu^i)}. \quad (43)$$

If we group lecture hall partitions according to the size of the last part as follows:

$$\mathbf{L}_n = \bigcup_{t \geq 0} \left\{ \lambda \in \mathbf{L}_n \mid t-1 < \frac{\lambda_n}{n} \leq t \right\}$$

the union is disjoint and applying Theorem 9.3 with $i = 0$ we have

$$\sum_{\lambda \in \mathbf{L}_n} x^{\lceil \lambda_n/n \rceil} u^{|\lambda|} = \sum_{t \geq 0} (([t+1]_u)^n - ([t]_u)^n) x^t = (1-x) \sum_{t \geq 0} ([t+1]_u)^n x^t. \quad (44)$$

Combining (43), and (44) gives the following.

Theorem 10.2 (S, Schuster [43]).

$$\sum_{\lambda \in \mathbf{L}_n} x^{\lceil \lambda_n/n \rceil} u^{|\lambda|} = \frac{\sum_{\pi \in \mathbf{S}_n} x^{\text{des } \pi} u^{\text{maj } \pi}}{\prod_{i=1}^n (1 - xu^i)}. \quad (45)$$

We show in Section 14 that Theorem 10.2 can be further refined and can be extended to describe statistics on \mathbf{s} -lecture hall partitions.

What statistic on permutations corresponds to the weight statistic $|\lambda|$ on lecture hall partitions? We consider this in the next section and again in Section 14.

11 Quadratic permutation statistics

Returning to the observation (34) in Section 8, with Bright, in [15] we found bijections $\mathbf{L}_n \rightarrow \mathbb{Z}^n$ and $\mathbf{A}_n \rightarrow \mathbb{Z}^n$ which, when combined, proved that $L_n(q, u) = A_n(q^{-1}, uq^{n+1})$.

In doing so we connected both lecture hall partitions and anti-lecture hall compositions to permutations and discovered some interesting statistics. The inversion number, $\text{inv } \pi$ of a permutation $\pi \in \mathbf{S}_n$ is defined by

$$\text{inv } \pi = |\{(i, j) \mid i < j \text{ and } \pi_i > \pi_j\}|.$$

Over \mathbf{S}_n , inv has the same distribution as maj . Define new statistics $\text{bin } \pi$ and $\text{sq } \pi$ by

$$\text{bin } \pi = \sum_{i \in \text{Des } \pi} \binom{i+1}{2}; \quad \text{sq } \pi = \sum_{i \in \text{Des } \pi} i^2.$$

Both bin and sq are *quadratic* permutation statistics, in that they are defined by a sum of *quadratic* functions of i over descent positions i .

It seems that quadratic statistics are not well-known, although they have appeared in the context of Weyl group generating functions ([53], [51], [57]). They show up in the following theorem.

Theorem 11.1 (Bright, S [15]).

$$A_n(q, u) = \sum_{\lambda \in \mathbf{A}_n} q^{|\lambda|} u^{|\lambda|} = \frac{\sum_{\pi \in \mathbf{S}_n} q^{\text{bin } \pi + \text{inv } \pi} u^{\text{maj } \pi}}{\prod_{i=1}^n (1 - u^i q^{i(i+1)/2})}.$$

This suggests merging bin and inv into a single statistic

$$\text{binv } \pi = \text{bin } \pi + \text{inv } \pi.$$

One can then combine Theorem 11.1 with Theorem 8.2 to compute the joint distribution of the quadratic statistic binv and the linear statistic maj over permutations.

Corollary 11.2 (Bright, S [15]).

$$\sum_{\pi \in \mathbf{S}_n} u^{\text{maj } \pi} q^{\text{binv } \pi} = \prod_{i=1}^n (1 - u^i q^{i(i+1)/2}) \frac{1 + uq^i}{1 - u^2 q^{i+1}}.$$

Setting $q = 1$ in Corollary 11.2 we get the well-known distribution of major index:

$$\sum_{\pi \in \mathbf{S}_n} u^{\text{maj } \pi} = \prod_{i=1}^n [i]_u.$$

Setting $u = 1$ in Corollary 11.2 shows that binv itself has an interesting generating function.

$$\sum_{\pi \in \mathbf{S}_n} q^{\text{binv } \pi} = \prod_{i=1}^n [2]_{q^i} \frac{[i(i+1)/2]_q}{[i+1]_q}.$$

Replacing q by q^{-1} and then setting $u = q^{n+1}$ in Corollary 11.2 gives a nice result:

$$\sum_{\pi \in \mathbf{S}_n} q^{(n+1)\text{maj } \pi - \text{binv } \pi} = \prod_{i=1}^n [i]_{q^{2(n-i)+1}} \quad (46)$$

and this product will appear again in Section 14. We get an even simpler joint distribution with maj if we consider the statistic

$$\text{sqin } \pi = \text{sq } \pi + \text{inv } \pi.$$

The following was given a simple combinatorial proof in [15] which, unlike Corollary 11.2, did not require a refined lecture hall theorem.

Theorem 11.3 (Bright, S [15]).

$$\sum_{\pi \in \mathbf{S}_n} u^{\text{maj } \pi} q^{\text{sqin } \pi} = \prod_{i=1}^n [i]_{uq^i}.$$

These distributions seem to have gone mostly unnoticed although the special case of Theorem 11.3 when $q = q^{-1}$ and $u = q^n$ was proven by Zabrocki in [57]. Recently, quadratic permutation statistics arose in work of Paul Johnson on q -rational Catalan numbers [35]. He required the following identity, which he proved in [35].

Corollary 11.4 (Johnson [35]).

$$\sum_{\pi \in \mathbf{S}_n} u^{\text{maj } \pi} q^{\text{siz } \pi} = \prod_{i=1}^n [i]_{uq^{n+1-i}},$$

where $\text{siz } \pi = (n+1)\text{maj } \pi - \text{sqin } \pi$.

Note that replacing q by q^{-1} and then setting $u = uq^{n+1}$ in Theorem 11.3 gives Corollary 11.4.

12 Lattice point generating functions of lecture hall cones

For a sequence \mathbf{s} of positive integers, the \mathbf{s} -lecture hall cone is defined as

$$\mathbf{C}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}. \quad (47)$$

Then $\mathbf{L}_n^{(\mathbf{s})} = \mathbf{C}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$. The lattice point generating function of $\mathbf{C}_n^{(\mathbf{s})}$ is

$$F_n^{(\mathbf{s})}(z) = \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} z^\lambda,$$

where $z^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_n^{\lambda_n}$. $L_n^{(\mathbf{s})}(q)$ is obtained by setting each $z_i = q$.

The cone $\mathbf{C}_n^{(\mathbf{s})}$ is generated by the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ where $\mathbf{v}_i = [0, \dots, 0, s_i, \dots, s_n]$:

$$\mathbf{C}_n^{(\mathbf{s})} = \left\{ \sum_{i=1}^n \alpha_i \mathbf{v}_i \mid \alpha_i \geq 0 \right\}.$$

The (half open) fundamental parallelepiped associated with $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is

$$\Pi_n^{(\mathbf{s})} = \left\{ \sum_{i=1}^n \alpha_i \mathbf{v}_i \mid 0 \leq \alpha_i < 1 \right\}.$$

It is well known (see for example [40] p. 40) that $F_n^{(\mathbf{s})}(z)$ can be expressed as follows.

Theorem 12.1.

$$F_n^{(\mathbf{s})}(z) = \frac{\sum_{\lambda \in \Pi_n^{(\mathbf{s})} \cap \mathbb{Z}^n} z^\lambda}{\prod_{i=1}^n (1 - z^{\mathbf{v}_i})}.$$

So, enumerating all \mathbf{s} -lecture hall partitions reduces to enumerating those in $\Pi_n^{(\mathbf{s})}$. There are $s_1 s_2 \cdots s_n$ lattice points in $\Pi_n^{(\mathbf{s})}$ and we will see in Section 14 that they can be characterized in terms of inversion sequences and their statistics.

Example: $\mathbf{C}_2^{(2,3)}$ has generators $[2, 3]$ and $[0, 3]$ and

$$\Pi_2^{(\mathbf{s})} \cap \mathbb{Z}^n = \{(0, 0), (0, 1), (0, 2), (1, 2), (1, 3), (1, 4)\},$$

so

$$F_2^{(2,3)}(z) = \frac{1 + z_2 + z_2^2 + z_1 z_2^2 + z_1 z_2^3 + z_1 z_2^4}{(1 - z_1^2 z_2^3)(1 - z_2^3)} = \frac{1 + z_1 z_2^2}{(1 - z_1^2 z_2^3)(1 - z_2)}$$

and

$$L_2^{(2,3)}(q) = \frac{1 + q^3}{(1 - q)(1 - q^5)}.$$

Note that we could replace the vector $\mathbf{v}_n = [0, 0, \dots, s_n]$ by $\mathbf{v}'_n = [0, 0, \dots, 1]$. This would generate the same cone, but the fundamental parallelepiped would be different. More about this in Section 18.

13 Lecture hall polytopes and Ehrhart theory

In Section 10 statistics on lecture hall partitions are described in terms of statistics on permutations. We show that it is possible to describe \mathbf{s} -lecture hall partitions in terms of statistics on a generalization of permutations.

The *s-lecture hall polytope* is the bounded region of the *s-lecture hall cone* defined by

$$\mathbf{P}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}. \quad (48)$$

The *t*-th dilation of $\mathbf{P}_n^{(\mathbf{s})}$ is $t\mathbf{P}_n^{(\mathbf{s})} = \{t\lambda \mid \lambda \in \mathbf{P}_n^{(\mathbf{s})}\}$.

The *Ehrhart polynomial* of $\mathbf{P}_n^{(\mathbf{s})}$ is the number of lattice points in its *t*-th dilation:

$$i_n^{(\mathbf{s})}(t) = |t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|.$$

The vertices of $\mathbf{P}_n^{(\mathbf{s})}$ are $\{(0, 0, \dots, 0, s_i, s_{i+1}, \dots, s_n \mid 1 \leq i \leq n+1\}$ and since all coordinates are integers, $i_n^{(\mathbf{s})}(t)$ is guaranteed to be a polynomial in *t* [28, 29].

The *Ehrhart series* of $\mathbf{P}_n^{(\mathbf{s})}$ is $\sum_{t \geq 0} i_n^{(\mathbf{s})}(t)x^t$ and it is known that

$$\sum_{t \geq 0} i_n^{(\mathbf{s})}(t)x^t = \frac{E_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}, \quad (49)$$

where $E_n^{(\mathbf{s})}(x)$ is a polynomial with nonnegative integer coefficients (see, e.g., [50]). $E_n^{(\mathbf{s})}(x)$ is called the *h*-polynomial* of the polytope $\mathbf{P}_n^{(\mathbf{s})}$.

The relationship between the \mathbf{s} -lecture hall partitions and $E_n^{(\mathbf{s})}(x)$, analogous to the derivation of (44) in Section 10, is then given by

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lceil \lambda_n / s_n \rceil} = (1-x) \sum_{t \geq 0} i_n^{(\mathbf{s})}(t)x^t = \frac{E_n^{(\mathbf{s})}(x)}{(1-x)^n}. \quad (50)$$

When $\mathbf{s} = (1, 2, \dots, n)$, by (37),

$$i_n^{(1,2,\dots,n)}(t) = (t+1)^n$$

and thus by (41) and (49), $E_n^{(1,2,\dots,n)}(x)$ is the Eulerian polynomial $E_n(x)$.

For this reason, we refer to $E_n^{(\mathbf{s})}(x)$ the *\mathbf{s} -Eulerian polynomial*. In the next section we provide a combinatorial interpretation of $E_n^{(\mathbf{s})}(x)$ for general \mathbf{s} .

14 Generalized lecture hall partitions and \mathbf{s} -inversion sequences

A permutation $\pi \in \mathbf{S}_n$ can be encoded as the *inversion sequence* $\phi(\pi) = (e_1, \dots, e_n)$ where

$$e_i = |\{j \mid j < i \text{ and } \pi_j > \pi_i\}|.$$

Then $i \in \text{Des}(\pi)$ if and only if $e_i < e_{i+1}$ and $\phi : \mathbf{S}_n \rightarrow \{(e_1, \dots, e_n) \mid 0 \leq e_i < i\}$ is a bijection. Inversion sequences will be used to generalize permutations.

Given a sequence of positive integers \mathbf{s} , define the *\mathbf{s} -inversion sequences* by

$$\mathbf{I}_n^{(\mathbf{s})} = \{\mathbf{e} \in \mathbb{Z}^n \mid 0 \leq e_i < s_i\}.$$

For $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$ define the *ascent set* of \mathbf{e} by

$$\text{Asc } \mathbf{e} = \left\{ i \mid 0 \leq i < n \text{ and } \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention that $e_0 = 0$ and $s_0 = 1$. Let $\text{asc } \mathbf{e} = |\text{Asc } \mathbf{e}|$.

So, for example, $e = (3, 2) \in \mathbf{I}_2^{(5,3)}$ has $\text{Asc}(3, 2) = \{0, 1\}$ since $0 < e_1/s_1 = 3/5$ and $e_1/s_1 = 3/5 < 2/3 = e_2/s_2$.

In view of (50), the following theorem provides a combinatorial characterization of the \mathbf{s} -Eulerian polynomial.

Theorem 14.1 (S, Schuster [43]). *For any sequence \mathbf{s} of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lceil \lambda_n / s_n \rceil} = \frac{\sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}}}{(1-x)^n}. \quad (51)$$

Proof. We outline the proof of Theorem 14.1 from [43], which adapts the idea of *barred permutations* from Gessel and Stanley [33].

A *barred inversion sequence* is an inversion sequence $\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}$ with “bars” between consecutive elements. It can have any number of bars before e_1 or between e_i and e_{i+1} , but if $i \in \text{Asc } \mathbf{e}$, at least one bar must occur between e_i and e_{i+1} .

We show that both sides of (51) count all barred inversion sequences of $\mathbf{I}_n^{(s)}$ according to the number of bars.

First, fix the inversion sequence $\mathbf{e} \in \mathbf{I}_n^{(s)}$ and consider all possible barrings of \mathbf{e} . There must be at least $\text{asc } \mathbf{e}$ bars and then any number of additional bars can be distributed into the n possible positions. Summing over all $\mathbf{e} \in \mathbf{I}_n^{(s)}$ gives the right-hand side of (51).

For the second way, we show the following is a bijection between lecture hall partitions with $\lceil \lambda_n/s_n \rceil = t$ and barred inversion sequences with t bars. Let $\lambda \in \mathbf{L}_n^{(s)}$ with $\lceil \lambda_n/s_n \rceil = t$. Let

$$b = (\lceil \lambda_1/s_1 \rceil, \lceil \lambda_2/s_2 \rceil, \dots, \lceil \lambda_n/s_n \rceil).$$

Then $b_1 \leq b_2 \leq \dots \leq b_n = t$. Let $\mathbf{e} = (e_1, \dots, e_n)$, where

$$e_i = s_i b_i - \lambda_i.$$

Clearly, $\mathbf{e} \in \mathbf{I}_n^{(s)}$, since $0 \leq e_i < s_i$. Insert t bars into \mathbf{e} by placing b_1 bars before e_1 and $b_i - b_{i-1}$ bars before e_i for $2 \leq i \leq n$. It can be checked that since $\lambda \in \mathbf{L}_n^{(s)}$, if there is no bar between e_i and e_{i+1} then $b_i = b_{i+1}$ and $e_i/s_i \geq e_{i+1}/s_{i+1}$ so that $i \notin \text{Asc } \mathbf{e}$.

Then summing the lecture hall partitions with $\lceil \lambda_n/s_n \rceil = t$ over all t , as in the left-hand side of (51) also gives the number of barred inversion sequences. \square

From Theorem 14.1 and (50) we have a characterization of the \mathbf{s} -Eulerian polynomials.

Corollary 14.2. *For any sequence \mathbf{s} of positive integers \mathbf{s}*

$$E_n^{(s)}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(s)}} x^{\text{asc } \mathbf{e}}.$$

That is, the h^ -polynomial of the s -lecture hall polytope is the ascent polynomial for the \mathbf{s} -inversion sequences.*

There is more about the s -Eulerian polynomials in the next section, but we note now that Theorem 14.1 can be refined to track other statistics. Define the following statistics on \mathbf{s} -inversion sequences \mathbf{e} :

$$\text{amaj } \mathbf{e} = \sum_{i \in \text{Asc } \mathbf{e}} (n - i)$$

$$\text{lhpe } \mathbf{e} = -|\mathbf{e}| + \sum_{i \in \text{Asc } \mathbf{e}} (s_{i+1} + \dots + s_n).$$

Define the following statistics on \mathbf{s} -lecture hall partitions λ :

$$\lceil \lambda \rceil = [\lceil \lambda_1/s_1 \rceil, \dots, \lceil \lambda_n/s_n \rceil]$$

$$\epsilon^+(\lambda) = [s_1 \lceil \lambda_1/s_1 \rceil - \lambda_1, \dots, s_n \lceil \lambda_n/s_n \rceil - \lambda_n]$$

Following the statistics on a lecture hall partition as it maps to a barred inversion sequence in the proof of Theorem 14.1 will give this 4-parameter refinement.

Theorem 14.3 (S, Schuster [43]). *For any sequence \mathbf{s} of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} x^{\lceil \lambda_n / s_n \rceil} u^{|\lceil \lambda \rceil} z^{|\epsilon^+(\lambda)|} = \frac{\sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}} u^{\text{amaj } \mathbf{e}} q^{\text{lhpe } \mathbf{e}} z^{|\mathbf{e}|}}{\prod_{i=0}^{n-1} (1 - xu^{n-i} q^{s_{i+1} + \dots + s_n})}.$$

In [43], Theorem 14.3 is applied in various ways to obtain known and new enumeration results. We mention just one here.

Setting $\mathbf{s} = (1, \dots, n)$ gives the following connection between lecture hall partitions and permutations, extending the results in Section 10.

Corollary 14.4.

$$\sum_{\lambda \in \mathbf{L}_n} q^{|\lambda|} u^{|\lambda|} x^{\lceil \lambda_n / n \rceil} z^{|\mathbf{e}|} = \frac{\sum_{\pi \in \mathbf{S}_n} x^{\text{des } \pi} u^{\text{comaj } \pi} q^{\text{lhpe } \pi} z^{\text{inv } \pi}}{\prod_{i=0}^{n-1} (1 - xu^{n-i} q^{(i+1) + \dots + n})},$$

where for $\pi \in \mathbf{S}_n$,

$$\text{comaj } \pi = \sum_{i \in \text{Des } \pi} (n - i)$$

and

$$\text{lhpe } \pi = -\text{inv } \pi + \sum_{i \in \text{Des } \pi} ((i + 1) + \dots + n).$$

The statistic lhpe in Corollary 14.4 is another appearance of a quadratic permutation statistic. As shown in [43], setting $x = z = 1$ in Corollary 14.4 and combining with the refined lecture hall theorem gives the joint distribution of lhpe and comaj :

Corollary 14.5 (S, Schuster [43]).

$$\sum_{\pi \in \mathbf{S}_n} q^{\text{lhpe } \pi} u^{\text{comaj } \pi} = \prod_{k=1}^n \frac{(1 + uq^k)(1 - u^{n+1-k} q^{k + \dots + n})}{1 - u^2 q^{n+k}}.$$

Setting $u = 1$ gives a simple generating polynomial for the lecture hall statistic on permutations, the same as (46).

Corollary 14.6 (S, Schuster [43]).

$$\sum_{\pi \in \mathbf{S}_n} q^{\text{lhpe } \pi} = \prod_{k=1}^n [k]_{q^{2(n-k)+1}}.$$

Note that by Theorem 14.3, Corollary 14.5 is *equivalent* to the refined lecture hall theorem and Corollary 14.6 is equivalent to the lecture hall theorem. Is there a simple combinatorial proof of Corollary 14.6?

	sequence \mathbf{s}	\mathbf{s} -Eulerian polynomial
(i)	(1, 2, 3, 4, 5, 6)	$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$
(ii)	(6, 5, 4, 3, 2, 1)	$1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$
(iii)	(2, 4, 6, 8, 10)	$1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5$
(iv)	(1, 3, 5, 7, 9, 11)	$1 + 358x + 3580x^2 + 5168x^3 + 1328x^4 + 32x^5$.
(v)	(1, 4, 3, 8, 5, 12)	$27x^5 + 811x^4 + 2828x^3 + 1884x^2 + 209x + 1$
(vi)	(1, 1, 3, 2, 5, 3)	$1 + 20x + 48x^2 + 20x^3 + x^4$
(vii)	(7, 2, 3, 5, 4, 6)	$1 + 71x + 948x^2 + 2450x^3 + 1411x^4 + 159x^5$

Table 1: \mathbf{s} -Eulerian polynomials

15 \mathbf{s} -Eulerian polynomials

By Corollary 14.2 in the previous section, $E_n^{(\mathbf{s})}(x)$ is the ascent polynomial of the \mathbf{s} -inversion sequences. Table 1 shows $E_n^{(\mathbf{s})}(x)$ for various sequences \mathbf{s} .

Row (i) of the table contains the Eulerian polynomial $E_6(x)$. The sequences in rows (i) and (ii) of the table give rise to the same Eulerian polynomial. This is true in general.

Theorem 15.1 (S, Schuster [43]). *For any sequence \mathbf{s} of positive integers,*

$$E_n^{(s_1, s_2, \dots, s_n)}(x) = E_n^{(s_n, s_{n-1}, \dots, s_1)}(x).$$

Proof. From Theorem 9.2, $\mathbf{P}_n^{(s_1, s_2, \dots, s_n)}$ and $\mathbf{P}_n^{(s_n, s_{n-1}, \dots, s_1)}$ have the same Ehrhart polynomial, so the result follows from (49). \square

The polynomial in row (iii) of Table 1 might be recognized as the Eulerian polynomial for *signed permutations*: Let $B_n = \{(\sigma_1, \dots, \sigma_n) \mid \exists \pi \in \mathbf{S}_n, \forall i \sigma_i = \pm \pi(i)\}$ and let $\text{des } \sigma = |\{i \in \{0, \dots, n-1\} \mid \sigma_i > \sigma_{i+1}\}|$, with the convention that $\sigma_0 = 0$.

Theorem 15.2 (Pensyl, S [42]).

$$\sum_{\sigma \in B_n} x^{\text{des } \sigma} = E_n^{(2,4,\dots,2n)}(x).$$

This was proved bijectively in [42] where it was refined to include Mahonian and other statistics. In fact, the idea extends to the wreath product $\mathbf{S}_n \wr \mathbb{Z}_k$ with the natural notion of descent.

Theorem 15.3 (Pensyl, S [41]).

$$\sum_{\sigma \in \mathbf{S}_n \wr \mathbb{Z}_k} x^{\text{des } \sigma} = E_n^{(k,2k,\dots,nk)}(x).$$

Observe that for $k \geq 1$, $\mathbf{L}_n^{(k,2k,\dots,nk)} = \mathbf{L}_n$. Thus by Theorems 14.1 and 15.3,

$$\sum_{\lambda \in \mathbf{L}_n} x^{[\lambda_n/(nk)]} = \frac{\sum_{\sigma \in \mathbf{S}_n \wr \mathbb{Z}_k} x^{\text{des } \sigma}}{(1-x)^n}.$$

We highlight this surprising result below.

Theorem 15.4 (Pensyl, S [41]). *The distribution of the descent statistic on the wreath product $\mathbf{S}_n \wr \mathbb{Z}_k$ is given by the distribution of the statistic $[\lambda_n/(nk)]$ over lecture hall partitions $\lambda_n \in \mathbf{L}_n$ (multiplied by a factor of $(1-x)^n$.) This includes permutations, when $k = 1$, and signed permutations, when $k = 2$.*

The $1/k$ -Eulerian polynomial $E_{n,k}(x)$ of [46] is the ascent polynomial of the “1 mod k ” inversion sequences, that is:

$$E_{n,k}(x) = E_n^{(1,k+1,2k+1,\dots,(n-1)k+1)}(x).$$

In Table 1, the polynomial in row (iv) is $E_{6,2}(x) = E_6^{(1,3,5,7,9,11)}(x)$. The $1/k$ -Eulerian polynomial derives its name from the following.

Theorem 15.5 (S, Viswanathan [46]). *For $k \geq 1$,*

$$\sum_{n \geq 0} E_{n,k}(x) \frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)} - x} \right)^{\frac{1}{k}}.$$

The $1/k$ -Eulerian polynomials are related to permutations as follows.

Theorem 15.6 (S, Viswanathan [46]).

$$E_{n,k}(x) = \sum_{e \in \mathbf{I}_n^{(1,k+1,2k+1,\dots,(n-1)k+1)}} x^{\text{asc } e} = \sum_{\pi \in \mathbf{S}_n} x^{\text{exc } \pi} k^{n - \text{cyc } \pi},$$

where $\text{exc } \pi = |\{i \mid \pi(i) > i\}|$ and $\text{cyc } \pi$ is the number of cycles in the disjoint cycle representation of π .

The number of 1 mod k -inversion sequences is $|\mathbf{I}_n^{(1,k+1,2k+1,\dots,(n-1)k+1)}| = \prod_{i=0}^{n-1} (ik + 1)$, which is the same as the number of k -Stirling permutations of order n . It is natural to ask if there is a statistic for k -Stirling permutations whose distribution is $E_{n,k}(x)$. In [38], Ma and Mansour show that $E_{n,k}(x)$ gives the distribution of a statistic called “ascent plateaus” on the k -Stirling permutations of order n . We are not aware of a bijective proof for this result or for Theorem 15.6.

The sequence in row (v) of Table 1 is the $(4,1)$ -sequence discussed in Section 7. Although the cardinality of $\mathbf{I}_{2n}^{(1,1,3,2,\dots,2n-1,n)}$ is the same as the number of permutations of $\{1, 1, 2, 2, \dots, n, n\}$, it comes as a surprise that the ascent generating function for the first set is the descent polynomial for the second.

Theorem 15.7 (S, Visontai [45]). $E_{2n}^{(1,1,3,2,\dots,2n-1,n)}(x)$ is the descent polynomial for permutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$.

To prove Theorem 15.7, we made use of the Ehrhart polynomial of $\mathbf{P}_{2n}^{(1,1,3,2,\dots,2n-1,n)}$, which had been computed in [43]. We then used (49) to get an explicit expression for $E_{2n}^{(1,1,3,2,\dots,2n-1,n)}(x)$. This matched MacMahon’s generating function for the descent polynomial of multiset permutations. A combinatorial proof would be quite interesting.

The sequence in Table 1, row (vi) is the $(1,4)$ -sequence. The following was conjectured in [45], and was proved independently by Lin in [36] and by Chen et al in [17].

Theorem 15.8 (Lin [36] and Chen et al [17]). $E_{2n}^{(1,4,3,8,\dots,2n-1,4n)}(x)$ is the descent polynomial for the signed permutations of $\{1, 1, 2, 2, \dots, n, n\}$.

16 Real-rooted polynomials

The example in the last row of Table 1 illustrates that the coefficient sequence of $E_n^{(\mathbf{s})}(x)$ can be unimodal, even though the sequence \mathbf{s} is not. Recall that $E_n^{(\mathbf{s})}(x)$ is the h^* -polynomial of the polytope $\mathbf{P}_n^{(\mathbf{s})}$. However, the coefficient sequence of the h^* -polynomial of a convex lattice polytope need not be unimodal. It turns out that $\mathbf{P}_n^{(\mathbf{s})}$ is special.

It is known that if a polynomial has all real roots, then its sequence of coefficients is unimodal and log-concave. In [45] we proved the following with Visontai, using the method of compatible polynomials developed by Chudnovsky and Seymour in [18].

Theorem 16.1 (S, Visontai [45]). *For any sequence \mathbf{s} of positive integers, the \mathbf{s} -Eulerian polynomial has only real roots.*

It then follows as a corollary of Theorem 16.1 that all of the polynomials discussed in the previous section are real-rooted. It has been known since Frobenius [32] that the Eulerian polynomials are real-rooted. It was first shown by Brenti that the type-B Eulerian polynomials have only real roots [13]. The real-rootedness of the descent polynomials of the wreath products $\mathbf{S}_n \wr \mathbb{Z}_k$ was shown by Steingrímsson [52]. In [14] Brenti proved that the polynomial $\sum_{\pi \in \mathbf{S}_n} x^{\text{exc } \pi} y^{\text{cyc } \pi}$ has all real roots for any positive y [14]. This implies that the $1/k$ -Eulerian polynomials have only real roots. Simion [48] established the real-rootedness of descent polynomials of multiset permutations.

In [13], Brenti conjectured that the descent polynomial of any finite Coxeter group has only real roots. This was known to be true for all except the type- D Coxeter groups, D_n . D_n consists of signed permutations in B_n in which an even number of elements are negative. Descents in D_n are defined as follows, but now with the convention that $\sigma_0 = -\sigma_2$.

$$\text{Des}_D \sigma = \{i \mid 0 \leq i < n \text{ and } \sigma_i > \sigma_{i+1}\}.$$

In [45] with Visontai, we were able to use inversion sequences to settle the the type- D case and thereby settle Brenti’s conjecture.

Theorem 16.2 (S, Visontai [45]). *The Eulerian polynomials of type D have only real roots.*

To prove Theorem 16.2, it can be checked that the generating polynomial for the type- D descent statistic over B_n is twice the descent polynomial of D_n . By correspondingly defining a type- D ascent statistic for the $(2, 4, \dots, 2n)$ -inversion sequences, the technique of the proof of Theorem 16.1 can be modified to prove Theorem 16.2.

In [12], Brändén translates the proof of Theorem 16.2 in [45] into the language of Coxeter groups.

Inversion sequences were used in [45] to settle or make progress on several other conjectures, including the following. Dilks, Peterson, and Stembridge defined a notion of “affine descent” for an irreducible affine Weyl group W . The *affine Eulerian polynomial* of W is then the generating polynomial for this statistic over elements of W . They conjectured that the affine Eulerian polynomials for all finite Weyl groups are real-rooted. It was known for all but types B and D . It was settled for type B in [45] and Yang and Zhang settled it for type D in [54] to complete the proof.

Theorem 16.3 (Dilks, Petersen, Stembridge [27], S, Visontai [45], Yang, Zhang [54]). *The affine Eulerian polynomials for all finite Weyl groups are real-rooted.*

17 Generating lecture hall partitions by height

The Ehrhart series for the lecture hall polytope $\mathbf{P}_n^{(\mathbf{s})}$ gives rise to the Eulerian polynomials $E_n^{(\mathbf{s})}(x)$ which have been surprisingly useful at modeling combinatorial statistics. However, $E_n^{(\mathbf{s})}(x)$ is the generating polynomial for $\lambda \in \mathbf{L}_n^{(\mathbf{s})}(x)$ according to the unusual statistic $\lceil \lambda_n/s_n \rceil$. What about $\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lambda_n}$, which would seem more natural?

Instead of working with the lecture hall polytope $\mathbf{P}_n^{(\mathbf{s})}$ consider the rational lecture hall polytope in which the bounding condition is $\lambda_n \leq 1$ rather than $\lambda_n/s_n \leq 1$:

$$\mathbf{R}_n^{(\mathbf{s})} = \left\{ \lambda \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \dots \leq \frac{\lambda_n}{s_n} \leq \frac{1}{s_n} \right\}. \quad (52)$$

Then the height generating function for lecture hall partitions is given by

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lambda_n} = (1-x) \sum_{t \geq 0} |t\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n| x^t. \quad (53)$$

The function

$$t \rightarrow |t\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|$$

is no longer guaranteed to be a polynomial, as it was for $\mathbf{P}_n^{(\mathbf{s})}$, but rather a quasi-polynomial. The Ehrhart series will have the form

$$\sum_{t \geq 0} |t\mathbf{R}_n^{(\mathbf{s})} \cap \mathbb{Z}^n| x^t = \frac{Q_n^{(\mathbf{s})}(x)}{(1-x)(1-x^{s_n})^n},$$

where $Q_n^{(\mathbf{s})}(x)$ is a polynomial with nonnegative integer coefficients.

In [42], with Pensyl, we prove that $Q_n^{(\mathbf{s})}(x)$ has the following interpretation in terms of inversion sequences. This gives the height generating function for $\mathbf{L}_n^{(\mathbf{s})}$. $Q_n^{(\mathbf{s})}(x)$, called the *inflated s -Eulerian polynomial*, is an “inflated” version of $E_n^{(\mathbf{s})}(x)$.

Theorem 17.1 (Pensyl, S [42]). *For any sequence \mathbf{s} of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lambda_n} = \frac{Q_n^{(\mathbf{s})}(x)}{(1-x^{s_n})^n} = \frac{\sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{s_n \text{asc } \mathbf{e} - e_n}}{(1-x^{s_n})^n}.$$

In contrast to the \mathbf{s} -Eulerian polynomial, $Q_n^{(\mathbf{s})}(x)$ is not real-rooted. Is the coefficient sequence of $Q_n^{(\mathbf{s})}(x)$ unimodal for all positive integer sequences \mathbf{s} ? (It seems to be from our computations.)

We can get an explicit expression for the height generating function of \mathbf{L}_n .

Corollary 17.2 (Pensyl, S [42]). *For the original lecture hall partitions, \mathbf{L}_n ,*

$$\sum_{\lambda \in \mathbf{L}_n} x^{\lambda_n} = \frac{\sum_{\pi \in \mathbf{S}_n} x^{n \operatorname{des} \pi - n + \pi_n}}{(1 - x^n)^n} = (1 - x) \sum_{j \geq 0} \sum_{i=0}^{n-1} (j+1)^{n-i} (j+2)^i x^{jn+i}.$$

Proof. Set $\mathbf{s} = (1, \dots, n)$ in Theorem 17.1. For the first equality, note that the mapping $\phi : \mathbf{S}_n \rightarrow \mathbf{I}_n$ defined by $\phi(\pi) = (e_1, \dots, e_n)$, where $e_i = |\{j > 0 \mid j < i \text{ and } \pi_j > \pi_i\}|$, has the property that $\operatorname{des} \pi = \operatorname{asc} \phi(\pi)$ and $e_n = n - \pi_n$.

For the second equality, we have computed the Ehrhart quasi-polynomial in (36):

$$|(jn + i)\mathbf{R}_n \cap \mathbb{Z}^n| = (j+1)^{n-i} (j+2)^i$$

and so the result follows by (53). \square

The polynomials $\sum_{\pi \in \mathbf{S}_n} x^{n \operatorname{des} \pi + \pi_n}$ have appeared before, for example in work of Chung and Graham on inversion-descent polynomials [19], where the following is shown:

Theorem 17.3 (Chung, Graham [19]).

$$\frac{\sum_{\pi \in \mathbf{S}_n} x^{n \operatorname{des} \pi + \pi_n}}{1 + x + x^2 + \dots + x^{n-1}} = \sum_{\pi \in \mathbf{S}_{n-1}} x^{n \operatorname{des} \pi + \pi_{n-1}}.$$

Theorem 17.3 implies that $Q_n^{(1, \dots, n)}(x)$ is divisible by $1 + x + \dots + x^{n-1}$. It can be shown that a similar result holds for any \mathbf{s} and there is a nice combinatorial characterization of the quotient.

Theorem 17.4 (Auli, S [4]). *For any sequence \mathbf{s} of positive integers,*

$$\frac{Q_n^{(\mathbf{s})}(x)}{1 + x + \dots + x^{s_n - 1}} = \sum_{\mathbf{e} \in \mathbf{I}_{n-1}^{(\mathbf{s})}} x^{s_n \operatorname{asc} \mathbf{e} - \lfloor s_n e_{n-1} / s_{n-1} \rfloor}$$

Finally, we note that Theorem 17.1 has a refinement analogous to Theorem 14.3.

Theorem 17.5 (Pensyl, S [42]). *For any sequence \mathbf{s} of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} x^{\lambda_n} u^{|\lambda|} z^{|\epsilon^+(\lambda)|} = \frac{\sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} \mathbf{e} - e_n} u^{\operatorname{maj} \mathbf{e}} q^{\operatorname{lhpe}_z \mathbf{e}^{|\mathbf{e}|}}}{\prod_{i=0}^{n-1} (1 - x^{s_n} u^{n-i} q^{s_{i+1} + \dots + s_n})}.$$

18 Fundamental parallelepipeds

It is helpful to return to Section 12 and view the polynomials $E_n^{(\mathbf{s})}(x)$, $Q_n^{(\mathbf{s})}(x)$, and $Q_n^{(\mathbf{s})}(x)/(1+x+\dots+x^{s_n-1})$ in terms of fundamental parallelepipeds associated with generating sets of the lecture hall cone $\mathbf{C}_n^{(\mathbf{s})}(x)$.

Let $V_n(\mathbf{s}) = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be the generating set for $\mathbf{C}_n^{(\mathbf{s})}$ where $\mathbf{v}_i = [0, \dots, 0, s_i, \dots, s_n]$. Let $V'_n(\mathbf{s}) = \{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\} \cup \{[0, \dots, 0, 1]\}$, also a generating set for $\mathbf{C}_n^{(\mathbf{s})}$. Let $\Pi_n(\mathbf{s})$ and $\Pi'_n(\mathbf{s})$ be the (half open) fundamental parallelepipeds associated with $V_n(\mathbf{s})$ and $V'_n(\mathbf{s})$, respectively.

Liu and Stanley [37] first observed that under the bijection of Theorem 14.1, the inverse image of the minimally barred inversion sequences (which are counted by the numerator in the right-hand side of (51)) is exactly those $\lambda \in \Pi_n(\mathbf{s})$. So we have, by Theorem 14.1:

$$\sum_{\lambda \in \Pi_n(\mathbf{s}) \cap \mathbb{Z}^n} x^{\lceil \lambda_n / s_n \rceil} = E_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{\text{asc } \mathbf{e}}. \quad (54)$$

Setting $z_n = x$ and $z_1 = \dots = z_{n-1} = 1$ in Theorem 12.1 and comparing to Theorem 17.1 we have:

$$\sum_{\lambda \in \Pi_n(\mathbf{s}) \cap \mathbb{Z}^n} x^{\lambda_n} = Q_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e} \in \mathbf{I}_n^{(\mathbf{s})}} x^{s_n \text{asc } \mathbf{e} - e_n}. \quad (55)$$

On the other hand, if we consider the generating set $V'_n(\mathbf{s})$ and reinterpret Theorem 12.1 in conjunction with Theorem 17.4 we get:

$$\sum_{\lambda \in \Pi'_n(\mathbf{s}) \cap \mathbb{Z}^n} x^{\lambda_n} = \frac{Q_n^{(\mathbf{s})}(x)}{1+x+\dots+x^{s_n-1}} = \sum_{\mathbf{e} \in \mathbf{I}_{n-1}^{(\mathbf{s})}} x^{s_n \text{asc } \mathbf{e} - \lfloor s_n e_{n-1} / s_{n-1} \rfloor}. \quad (56)$$

Chung and Graham observed in [19] the following consequence of Theorem 17.3: the sequence of nonzero coefficients of $Q_n^{(1, \dots, n)}(x)/(1+x+\dots+x^{n-1})$ coincides with that of $Q_{n-1}^{(1, \dots, n-1)}(x)$. This generalizes as follows.

Theorem 18.1 (Auli, S [4]). *For every nondecreasing sequence \mathbf{s} of positive integers, the sequence of nonzero coefficients of $Q_n^{(\mathbf{s})}(x)/(1+x+\dots+x^{s_n-1})$ coincides with that of $Q_{n-1}^{(\mathbf{s})}(x)$.*

Note that for such \mathbf{s} , this means that the set $\Pi'_n(\mathbf{s}) \cap \mathbb{Z}^n$ can be constructed from $\Pi_{n-1}(\mathbf{s}) \cap \mathbb{Z}^{n-1}$ by lifting each $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ to $(\lambda_1, \dots, \lambda_{n-1}, \lfloor s_n \lambda_{n-1} / s_{n-1} \rfloor)$. And the set $\Pi_n(\mathbf{s}) \cap \mathbb{Z}^n$ can be constructed from the set $\Pi'_n(\mathbf{s}) \cap \mathbb{Z}^n$ by mapping each $\lambda = (\lambda_1, \dots, \lambda_n)$ to the s_n points $(\lambda_1, \dots, \lambda_n + i)$ for $0 \leq i < s_n$.

A complete characterization of such sequences is given in [4].

19 Gorenstein cones

A pointed rational cone $C \subseteq \mathbb{R}^n$ is *Gorenstein* if there exists a lattice point c in the interior C^0 of C such that

$$C^0 \cap \mathbb{Z}^n = c + (C \cap \mathbb{Z}^n).$$

For example, $\mathbf{C}_2^{(3,5)}$ is Gorenstein, but $\mathbf{C}_2^{(5,2)}$ is not.

A rational function $H(w_1, \dots, w_k)$ is *self-reciprocal* if

$$H(1/w_1, \dots, 1/w_k) = (\pm 1)w_1^{d_1} \dots w_k^{d_k} H(w_1, \dots, w_k)$$

for some integers d_1, \dots, d_k .

A wonderful property of Gorenstein cones is that the generating function for its lattice points according to *any* “proper” grading is self-reciprocal. A *proper grading* of C is a function $g : C \rightarrow \mathbb{N}^r$, for some r , satisfying (i) $g(\lambda + \mu) = g(\lambda) + g(\mu)$; (ii) $g(\lambda) = 0$ implies $\lambda = 0$; and (iii) for any $v \in \mathbb{N}^r$, $g^{-1}(v)$ is finite.

Examples of proper gradings on $\mathbf{C}_n^{(\mathbf{s})}$ are

- $\lambda \rightarrow (\lambda_1, \dots, \lambda_n)$
- $\lambda \rightarrow \lambda_n$
- $\lambda \rightarrow |\lambda|$.

However, for general \mathbf{s} , $\lambda \rightarrow \lceil \lambda \rceil$ is not proper nor is $\lambda \rightarrow \lceil \lambda_n / s_n \rceil$.

For $m = (m_1, \dots, m_r) \in \mathbb{N}^r$ let $X^m = X_1^{m_1} \dots X_r^{m_r}$. The following is a special case of a result due to Stanley [49].

Theorem 19.1 (Stanley [49]). *Let $g : C \rightarrow \mathbb{N}^r$ be a proper grading of a pointed rational cone C and let H be the generating function*

$$H(X_1, \dots, X_r) = H(X) = \sum_{\lambda \in C} X^{g(\lambda)}.$$

Then C is Gorenstein if and only if H is self-reciprocal.

So in Gorenstein lecture hall cones $\mathbf{C}_n^{(\mathbf{s})}(x)$, all of the following are self-reciprocal:

- The lattice point generating function $F_n^{(\mathbf{s})}(z)$.
- The generating function $L_n^{(\mathbf{s})}(q)$.
- The Ehrhart series of the polytope $\mathbf{R}_n^{(\mathbf{s})}$ (but not necessarily of $\mathbf{P}_n^{(\mathbf{s})}$).

- The inflated \mathbf{s} -Eulerian polynomial $Q_n^{(\mathbf{s})}(x)$ (but not necessarily the \mathbf{s} -Eulerian polynomial $E_n^{(\mathbf{s})}(x)$).
- The polynomial $Q_n^{(\mathbf{s})}(x)/(1+x+\dots+x^{s_n-1})$.

And if the cone is not Gorenstein then none of these can be self-reciprocal.

So when are lecture hall cones Gorenstein? This brings us back to Section 2 and the original question of Bousquet-Mélou and Eriksson [9] regarding polynomic sequences: Which sequences \mathbf{s} have $L_n^{(\mathbf{s})}(q) = \prod_{i=1}^n (1 - q^{d_i})^{-1}$? Clearly, in order for \mathbf{s} to be polynomic, $L_n^{(\mathbf{s})}(q)$ must be self-reciprocal, as Bousquet-Mélou and Eriksson pointed out, and therefore by Theorem 19.1 the cone must be Gorenstein. And by Theorem 19.1 this is guaranteed if the lattice point generating function $F_n^{(\mathbf{s})}(z)$ is self-reciprocal.

Gorenstein \mathbf{s} -lecture hall cones can be characterized by simple arithmetic conditions on the sequence \mathbf{s} .

Theorem 19.2 (Beck et al [6], Bousquet-Mélou, Eriksson [9]). *For any sequence \mathbf{s} of positive integers, $\mathbf{C}_n^{(\mathbf{s})}$ is Gorenstein if and only if there is a $c \in \mathbb{Z}^n$ satisfying*

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j-1})$$

for $j > 1$ with $c_1 = 1$.

The (k, ℓ) -sequences satisfy the conditions of Theorem 19.2 as do the sequences $(1, k + 1, 2k + 1, \dots)$. The Fibonacci sequence does not satisfy it for $n \geq 5$.

Beck et al showed in [6] that the ℓ -sequences are unique among sequences defined by second order linear recurrences in the following sense.

Theorem 19.3 (Beck et al [6]). *Let \mathbf{s} be a sequence of positive integers defined by*

$$s_n = \ell s_{n-1} + m s_{n-2},$$

with $s_1 = 1, s_2 = \ell$, where $\ell > 0$ and either $m > 0$ or $0 < |m| < \ell$. Then $\mathbf{C}_n^{(\mathbf{s})}$ is Gorenstein for all $n \geq 1$ if and only if $m = -1$. (That is, if and only if \mathbf{s} is an ℓ -sequence.) If $m \neq -1$, there exists $n_0 = n_0(\ell, m)$ such that $\mathbf{C}_n^{(\mathbf{s})}$ fails to be Gorenstein for all $n \geq n_0$.

As a consequence of Theorem 19.3, no other sequence defined by a second order linear recurrence of this form can be polynomic except perhaps for finitely many values of n .

The proof of Theorem 19.3 is quite involved when $\gcd(s_i, s_{i+1}) > 1$. Perhaps there is a simpler approach?

20 Concluding remarks

This report is by no means comprehensive, but rather gives a sampling of results, techniques, and connections in the developing theory of lecture hall partitions. A recent result, for example, shows that the lecture hall cone for \mathbf{L}_n has a unimodular triangulation [5]. The geometry has led to a lot of interesting results, observations, and connections. But there is still no transparent proof of the lecture hall theorem and, as yet, no geometric proof.

Acknowledgments I would like to thank all of my collaborators on the lecture hall projects. Thanks to AIM for hosting a SQuaRE in Polyhedral Geometry and Partition Theory, to the IMA for hosting a Workshop on Geometric and Enumerative Combinatorics, and to the Simons Foundation for travel support for collaboration. I am grateful to the referees for their corrections and suggestions to improve the manuscript.

I am particularly indebted to Herbert Wilf. I first learned about integer partitions at his invited address at the 1988 SIAM Discrete Math Conference in San Francisco. That changed everything.

References

- [1] George E. Andrews. MacMahon's partition analysis. I. The lecture hall partition theorem. In *Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996)*, volume 161 of *Progr. Math.*, pages 1–22. Birkhäuser Boston, Boston, MA, 1998.
- [2] George E. Andrews, Sylvie Corteel, and Carla D. Savage. On q -series identities arising from lecture hall partitions. *Int. J. Number Theory*, 5(2):327–337, 2009.
- [3] George E. Andrews, Peter Paule, Axel Riese, and Volker Strehl. MacMahon's partition analysis. V. Bijections, recursions, and magic squares. In *Algebraic combinatorics and applications (Gößweinstein, 1999)*, pages 1–39. Springer, Berlin, 2001.
- [4] Juan S. Auli and Carla D. Savage. Coefficients of the inflated Eulerian polynomial. 2015. preprint, arXiv:1504.01089 [math.NT].
- [5] Matthias Beck, Benjamin Braun, Matthias Köppe, Carla D. Savage, and Zafeirakis Zafeirakopoulos. Generating functions and triangulations for lecture hall cones. 2015. preprint, arXiv:1508.04619 [math.CO].
- [6] Matthias Beck, Benjamin Braun, Matthias Köppe, Carla D. Savage, and Zafeirakis Zafeirakopoulos. s -lecture hall partitions, self-reciprocal polynomials, and Gorenstein cones. *Ramanujan J.*, 36(1-2):123–147, 2015.
- [7] Alexander Berkovich and Ali Kemal Uncu. On partitions with fixed number of even-indexed and odd-indexed odd parts. *Journal of Number Theory*, 167:7–30, 2016.

- [8] Mireille Bousquet-Mélou and Kimmo Eriksson. Lecture hall partitions. *Ramanujan J.*, 1(1):101–111, 1997.
- [9] Mireille Bousquet-Mélou and Kimmo Eriksson. Lecture hall partitions. II. *Ramanujan J.*, 1(2):165–185, 1997.
- [10] Mireille Bousquet-Mélou and Kimmo Eriksson. A refinement of the lecture hall theorem. *J. Combin. Theory Ser. A*, 86(1):63–84, 1999.
- [11] Laura Bradford, Meredith Harris, Brant Jones, Alex Komarinski, Carly Matson, and Edwin O’Shea. The refined lecture hall theorem via abacus diagrams. *Ramanujan J.*, 34(2):163–176, 2014.
- [12] Petter Brändén. Unimodality, log-concavity, real-rootedness and beyond. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 437–483. CRC Press, Boca Raton, FL, 2015.
- [13] Francesco Brenti. q -Eulerian polynomials arising from Coxeter groups. *Eur. J. Comb.*, 15:417–441, September 1994.
- [14] Francesco Brenti. A class of q -symmetric functions arising from plethysm. *J. Combin. Theory Ser. A*, 91(1-2):137–170, 2000. In memory of Gian-Carlo Rota.
- [15] Katie L. Bright and Carla D. Savage. The geometry of lecture hall partitions and quadratic permutation statistics. In *22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010)*, Discrete Math. Theor. Comput. Sci. Proc., AN, pages 569–580. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.
- [16] William Y. C. Chen, Doris D. M. Sang, and Diane Y. H. Shi. Anti-lecture hall compositions and overpartitions. *J. Combin. Theory Ser. A*, 118(4):1451–1464, 2011.
- [17] William Y.C. Chen, Guo Alan J.X., Peter L. Guo, Harry H. Y. Huang, and Thomas Y.H. Liu. s -Inversion sequences and P -partitions of type B. 2013. preprint, arXiv:1310.5313 [math.CO].
- [18] Maria Chudnovsky and Paul Seymour. The roots of the independence polynomial of a clawfree graph. *J. Combin. Theory Ser. B*, 97(3):350–357, 2007.
- [19] Fan Chung and Ron Graham. Inversion-descent polynomials for restricted permutations. *J. Comb. Theory Ser. A*, 120(2):366–378, 2013.
- [20] Sylvie Corteel, Sunyoung Lee, and Carla D. Savage. Enumeration of sequences constrained by the ratio of consecutive parts. *Sém. Lothar. Combin.*, 54A:Art. B54Aa, 12, 2005/07.

- [21] Sylvie Corteel, Sunyoung Lee, and Carla D. Savage. Five guidelines for partition analysis with applications to lecture hall-type theorems. In *Combinatorial number theory*, pages 131–155. de Gruyter, Berlin, 2007.
- [22] Sylvie Corteel and Jeremy Lovejoy. Overpartitions. *Trans. Amer. Math. Soc.*, 356(4):1623–1635 (electronic), 2004.
- [23] Sylvie Corteel, Jeremy Lovejoy, and Carla Savage. Anti-lecture hall compositions and Andrews’ generalization of the Watson-Whipple transformation. *J. Combin. Theory Ser. A*, 134:188–195, 2015.
- [24] Sylvie Corteel and Carla D. Savage. Anti-lecture hall compositions. *Discrete Math.*, 263(1-3):275–280, 2003.
- [25] Sylvie Corteel and Carla D. Savage. Lecture hall theorems, q -series and truncated objects. *J. Combin. Theory Ser. A*, 108(2):217–245, 2004.
- [26] Sylvie Corteel, Carla D. Savage, and Andrew V. Sills. Lecture hall sequences, q -series, and asymmetric partition identities. In *Partitions, q -series, and modular forms*, volume 23 of *Dev. Math.*, pages 53–68. Springer, New York, 2012.
- [27] Kevin Dilks, T. Kyle Petersen, and John R. Stembridge. Affine descents and the Steinberg torus. *Adv. in Appl. Math.*, 42(4):423–444, 2009.
- [28] Eugene Ehrhart. Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux. *J. Reine Angew. Math.*, 226:1–29, 1967.
- [29] Eugene Ehrhart. Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires. *J. Reine Angew. Math.*, 227:25–49, 1967.
- [30] Alexander Engström and Matthew T. Stamps. Betti diagrams from graphs. *Algebra Number Theory*, 7(7):1725–1742, 2013.
- [31] Niklas Eriksen. A simple bijection between lecture hall partitions and partitions into odd integers. In *Formal Power Series and Algebraic Combinatorics*, 2002.
- [32] G. Frobenius. Über die Bernoullischen Zahlen und die Eulerschen polynome. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften*, 1910. Zweiter Halbband.
- [33] Ira Gessel and Richard P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
- [34] Heinz Göllnitz. *Partitionen mit Differenzenbedingungen*. Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Georg-August-Universität zu Göttingen. Dissertation, Göttingen, 1963.

- [35] Paul Johnson. Lattice points and simultaneous core partitions. 2015. preprint, arXiv:1502.07934 [math.NT].
- [36] Zhicong Lin. On the descent polynomial of signed multipermutations. *Proc. Amer. Math. Soc.*, 143(9):3671–3685, 2015.
- [37] Fu Liu and Richard P. Stanley. The lecture hall parallelepiped. *Ann. Comb.*, 18(3):473–488, 2014.
- [38] Shi-Mei Ma and Toufik Mansour. The $1/k$ -Eulerian polynomials and k -Stirling permutations. *Discrete Math.*, 338(8):1468–1472, 2015.
- [39] Percy A. MacMahon. *Combinatory analysis*. Two volumes (bound as one). Chelsea Publishing Co., New York, 1960.
- [40] Ezra Miller, Victor Reiner, and Bernd Sturmfels, editors. *Geometric combinatorics*, volume 13 of *IAS/Park City Mathematics Series*. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2007. Lectures from the Graduate Summer School held in Park City, UT, 2004.
- [41] Thomas W. Pensyl and Carla D. Savage. Lecture hall partitions and the wreath products $C_k \wr S_n$. *Integers*, 12B(Proceedings of the Integers Conference 2011):Paper No. A10, 18, 2012/13.
- [42] Thomas W. Pensyl and Carla D. Savage. Rational lecture hall polytopes and inflated Eulerian polynomials. *Ramanujan J.*, 31(1-2):97–114, 2013.
- [43] Carla D. Savage and Michael J. Schuster. Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences. *J. Combin. Theory Ser. A*, 119(4):850–870, 2012.
- [44] Carla D. Savage and Andrew V. Sills. On an identity of Gessel and Stanton and the new little Göllnitz identities. *Adv. in Appl. Math.*, 46(1-4):563–575, 2011.
- [45] Carla D. Savage and Mirkó Visontai. The s -Eulerian polynomials have only real roots. *Trans. Amer. Math. Soc.*, 367(2):1441–1466, 2015.
- [46] Carla D. Savage and Gopal Viswanathan. The $1/k$ -Eulerian polynomials. *Electron. J. Combin.*, 19(1):Paper 9, 21, 2012.
- [47] Carla D. Savage and Ae Ja Yee. Euler’s partition theorem and the combinatorics of ℓ -sequences. *J. Combin. Theory Ser. A*, 115(6):967–996, 2008.
- [48] Rodica Simion. A multi-indexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences. *J. Combin. Theory Ser. A*, 36(1):15–22, 1984.
- [49] Richard P. Stanley. Hilbert functions of graded algebras. *Advances in Math.*, 28(1):57–83, 1978.

- [50] Richard P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980. Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).
- [51] A. Stasinski and C. Voll. Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B . *Amer. J. Math.*, 136(2):501–550, 2014.
- [52] Einar Steingrímsson. *Permutations Statistics of Indexed and Poset Permutations*. PhD thesis, Massachusetts Institute of Technology, 1992.
- [53] John R. Stembridge and Debra J. Waugh. A Weyl group generating function that ought to be better known. *Indag. Math. (N.S.)*, 9(3):451–457, 1998.
- [54] Arthur L. B. Yang and Philip B. Zhang. Mutual interlacing and Eulerian-like polynomials for Weyl groups. 2014. preprint, arXiv:401.6273 [math.NT].
- [55] Ae Ja Yee. On the combinatorics of lecture hall partitions. *Ramanujan J.*, 5(3):247–262, 2001.
- [56] Ae Ja Yee. On the refined lecture hall theorem. *Discrete Math.*, 248(1-3):293–298, 2002.
- [57] Mike Zabrocki. A bijective proof of an unusual symmetric group generating function. 2003. preprint, arXiv:math/0910301v1.