Ehrhart series of lecture hall polytopes and Eulerian polynomials for inversion sequences

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Abstract

For a sequence $\mathbf{s} = (s_1, \ldots, s_n)$ of positive integers, an $\mathbf{s}$-*lecture hall partition* is an integer sequence λ satisfying $0 \leq \lambda_1/s_1 \leq \lambda_2/s_2 \leq \ldots \leq \lambda_n/s_n$. In this work, we introduce $\mathbf{s}$-*lecture hall polytopes*, $\mathbf{s}$-*inversion sequences*, and relevant statistics on both families. We show that for *any* sequence $\mathbf{s}$ of positive integers: (i) the h^*-vector of the $\mathbf{s}$-lecture hall polytope is the *ascent polynomial* for the associated $\mathbf{s}$-inversion sequences; (ii) the ascent polynomials for $\mathbf{s}$-inversion sequences generalize the Eulerian polynomials, including a q-analog that tracks a generalization of major index on $\mathbf{s}$-inversion sequences; and (iii) the generating function for the $\mathbf{s}$-lecture hall partitions can be interpreted in terms of a *new* q-analog of the $\mathbf{s}$-Eulerian polynomials, which tracks a "lecture hall" statistic on $\mathbf{s}$-inversion sequences. We show how four different statistics are related through the three $\mathbf{s}$-families of partitions, polytopes, and inversion sequences. Our approach uses Ehrhart theory to relate the partition theory of lecture hall partitions to their geometry.

Keywords: Lecture hall partitions, Eulerian polynomials, permutation statistics, Ehrhart-theory, inversion sequences, q-series identities

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1 Introduction

In this work, we establish a relationship between *Mahonian statistics*, such as $\mathrm{maj_A}$ and $\mathrm{inv_A}$ on permutations, *Eulerian statistics*, such as $\mathrm{des_A}$, and certain *lecture hall partition statistics.* We prove that this relationship extends beyond permutations and lecture hall partitions to the more general classes of $\mathbf{s}$-*inversion sequences* and $\mathbf{s}$-*lecture hall partitions.*

Throughout, we let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real numbers and integers, respectively. For $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$, the *weight* of p is $|p| = p_1 + \cdots + p_n$. If $p \in \mathbb{Z}$ and $0 \le p_1 \le \ldots \le p_n$, then p is a *partition* of $|p|$.

For a sequence $\mathbf{s} = \{s_i\}_{i\ge1}$ of positive integers, an $\mathbf{s}$-*lecture hall partition* is a finite integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n}.$$

Let $\mathbf{L}_n^{(\mathbf{s})}$ denote the set of all $\mathbf{s}$-lecture hall partitions of length n. For example:

- The elements of $\mathbf{L}_n^{(1,2,3,\ldots,n)}$ are the "original" *lecture hall partitions* introduced in 1997 by Bousquet-Mèlou and Eriksson [2].
- $\mathbf{L}_n^{(1,1,\ldots,1)}$ is the set of *integer partitions* into n nonnegative parts.
- $\mathbf{L}_n^{(n,n-1,\ldots,1)}$ is the set of *anti-lecture hall compositions* studied in [9].

Unless $\mathbf{s}$ is weakly increasing, an $\mathbf{s}$-lecture hall partition need not be a *partition.* For example, the anti-lecture hall composition $\lambda = (1, 2, 1) \in \mathbf{L}_n^{(3,2,1)}$ is not a partition.

Lecture hall partitions attracted immediate attention in 1997 because of their mysteriously simple generating function, discovered and proved by Bousquet-Mèlou and Eriksson in [2]:

$$\sum_{\lambda\in\mathbf{L}_n^{(1,2,\ldots,n)}} q^{|\lambda|} = \prod_{i=1}^{n} \frac{1}{1-q^{2i-1}}. \tag{1}$$

Since their discovery, lecture hall partitions and their generalizations have emerged as fundamental tools for interpreting classical partition identities and for discovering new ones [1, 2, 3, 4, 8, 9, 10, 12, 30, 31].

In this paper we introduce $\mathbf{s}$-*lecture hall polytopes* to establish a connection between the partition theory of $\mathbf{s}$-lecture hall partitions and their geometry. We introduce $\mathbf{s}$-*inversion sequences* to establish a connection between the geometry of lecture hall partitions and Mahonian, Eulerian, and lecture hall statistics on $\mathbf{s}$-inversion sequences.

In the next section, we review results about lecture hall partitions, permutation statistics and Eulerian polynomials; introduce $\mathbf{s}$-lecture hall polytopes and derive a recurrence for a refinement of their Ehrhart polynomial; introduce $\mathbf{s}$-inversion sequences and their statistics and establish their relationship with permutations when $\mathbf{s} = (1, 2, \ldots)$. Barred inversion sequences are defined in Section 2.6.

In Section 3, we prove our main results, Theorems 5, 6, and 7, relating

- the refined *Ehrhart polynomial* for the $\mathbf{s}$-lecture hall *polytope*;
- the joint distribution of *statistics* over $\mathbf{s}$-*inversion sequences*; and
- the multivariate *generating function* for the $\mathbf{s}$-*lecture hall partitions.*

As a consequence, for *any* sequence $\mathbf{s}$ of positive integers, we are now able to translate results in any one of these three domains into the others.

In Section 4, we demonstrate how the results of Section 3 can be applied to discover new relationships and to provide new context for known results. This sampling includes: the q-binomial theorem; the $\mathbf{s}$-Eulerian numbers; Eulerian polynomials and the lhp statistic; Mahonian statistics and lecture hall statistics; k-ary words and lecture hall statistics; signed Eulerian polynomials; interpreting q-series products with $\mathbf{s} = (1,2,1,2,\ldots)$; signed permutations and lecture hall polytopes; the sequences $(1,4,3,8,5,12,\ldots)$ and $(1,1,3,2,5,3,\ldots)$; the sequences $(1,k,k+1,2k+1,\ldots)$; and the ℓ-Eulerian polynomials.

Our main results integrate and extend previous work on lecture hall partitions and we highlight these connections in Section 5. Readers familiar with results on lecture hall partitions might look ahead.

2 Definitions and background

2.1 Generating functions for lecture hall partitions

We review some of the main results. In the enumeration of $\mathbf{s}$-lecture hall partitions, λ, there are two statistics of interest, in addition to the weight, $|\lambda|$. Define

$$\lceil \lambda \rceil = (\lceil \lambda_1/s_1 \rceil, \lceil \lambda_2/s_2 \rceil, \ldots, \lceil \lambda_n/s_n \rceil);$$
$$\lfloor \lambda \rfloor = (\lfloor \lambda_1/s_1 \rfloor, \lfloor \lambda_2/s_2 \rfloor, \ldots, \lfloor \lambda_n/s_n \rfloor).$$

Theorem 1. The Refined Lecture Hall Theorem [4]:

$$L_n(u,q) \triangleq \sum_{\lambda \in \mathbf{L}_n^{(1,2,\ldots,n)}} u^{|\lceil \lambda \rceil|} q^{|\lambda|} = \frac{(-uq;q)_n}{(u^2 q^{n+1};q)_n}, \tag{2}$$

where $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

Setting $u=1$ in (2) gives (1) as follows:

$$L_n(1,q) = \frac{(-q;q)_n}{(q^{n+1};q)_n} = \frac{(q^2;q^2)_n}{(q;q)_{2n}} = \frac{1}{(q;q^2)_n}.$$

Theorem 2. The Refined Anti-Lecture Hall Theorem [9]:

$$A_n(u,q) \triangleq \sum_{\lambda \in \mathbf{L}_n^{(n,n-1,\ldots,1)}} u^{|\lfloor \lambda \rfloor|} q^{|\lambda|} = \frac{(-uq;q)_n}{(u^2 q^2;q)_n}. \tag{3}$$

For what follows, we associate two additional sequences with an $\mathbf{s}$-lecture hall partition:

$$\epsilon^+(\lambda) \triangleq (s_1 \lceil \lambda_1/s_1 \rceil - \lambda_1,\ s_2 \lceil \lambda_2/s_2 \rceil - \lambda_2,\ \ldots,\ s_n \lceil \lambda_n/s_n \rceil - \lambda_n);$$
$$\epsilon^-(\lambda) \triangleq (\lambda_1 - s_1 \lfloor \lambda_1/s_1 \rfloor,\ \lambda_2 - s_2 \lfloor \lambda_2/s_2 \rfloor,\ \ldots,\ \lambda_n - s_n \lfloor \lambda_n/s_n \rfloor).$$

2.2 Lecture hall polytopes

For a sequence $\mathbf{s} = \{s_i\}_{i \geq 1}$ of positive integers, define the *lecture hall polytope* $\mathbf{P}_n^{(\mathbf{s})}$ by

$$\mathbf{P}_n^{(\mathbf{s})} \;=\; \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$

$\mathbf{P}_n^{(\mathbf{s})}$ is a convex, simplicial polytope with the following $n+1$ vertices:

$$\{(0, 0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n) \mid 1 \leq i \leq n+1\},$$

all with integer coordinates. The *t-th dilation of* $\mathbf{P}_n^{(\mathbf{s})}$ is the polytope

$$t\mathbf{P}_n^{(\mathbf{s})} \;=\; \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq t \right\}.$$

Let $\mathbf{f}_n^{(\mathbf{s})}(t)$ be the number of integer points in $t\mathbf{P}_n^{(\mathbf{s})}$:

$$\mathbf{f}_n^{(\mathbf{s})}(t) \;=\; |t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n|.$$

Since $\mathbf{P}_n^{(\mathbf{s})}$ is a convex polytope with integer vertices, $\mathbf{f}_n^{(\mathbf{s})}(t)$ is a polynomial in t of degree n, called the *Ehrhart polynomial* of $\mathbf{P}_n^{(\mathbf{s})}$ [13, 14]. Moreover,

$$\sum_{t \geq 0} \mathbf{f}_n^{(\mathbf{s})}(t) x^t \;=\; \frac{\mathbf{h}_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}, \tag{4}$$

where $\mathbf{h}_n^{(\mathbf{s})}(x)$ is a polynomial with *nonnegative* integer coefficients [26]. The degree of $\mathbf{h}_n^{(\mathbf{s})}(x)$ is the dimension of $\mathbf{P}_n^{(\mathbf{s})}$. The sequence of coefficients of $\mathbf{h}_n^{(\mathbf{s})}(x)$ is known as the $\mathbf{h}^*$-vector of $\mathbf{P}_n^{(\mathbf{s})}$ [26]. The infinite series defined in (4) is the *Ehrhart series* of $\mathbf{P}_n^{(\mathbf{s})}$.

We will be interested in the following refinement of $\mathbf{f}_n^{(\mathbf{s})}(t)$:

$$\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) \;=\; \sum_{\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap Z^n} u^{|\lceil \lambda \rceil|} q^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|}. \tag{5}$$

Note that

$$\mathbf{f}_n^{(\mathbf{s})}(t; 1, 1, 1) \;=\; \left| t\mathbf{P}_n^{(\mathbf{s})} \cap Z^n \right| \;=\; \mathbf{f}_n^{(\mathbf{s})}(t).$$

2.3 A recurrence for the refined Ehrhart polynomial

For positive integer sequence, $\mathbf{s}$, and nonnegative integers n, j, and i, with $0 \leq i \leq s_n$, let

$$\mathbf{L}_n^{(\mathbf{s};\, j,i)} \;=\; \{\lambda \in \mathbf{L}_n^{(\mathbf{s})} \mid \lambda_n \leq j s_n + i\}. \tag{6}$$

(Define $s_0 = 0$.) Let $L_n^{(\mathbf{s};\, j,i)}(u, q, z)$ be the generating function for $\mathbf{L}_n^{(\mathbf{s};\, j,i)}$, defined by

$$L_n^{(\mathbf{s};\, j,i)}(u, q, z) \;\triangleq\; \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s};\, j,i)}} u^{|\lceil \lambda \rceil|} q^{|\lambda|} z^{|\,\epsilon^+(\lambda)\,|}. \tag{7}$$

Theorem 3. (Refinement of Theorem 1 from [8]) *Let* $\mathbf{s}$ *be any sequence of positive integers. For integers* $n \geq 0$, $j \geq 0$, *and* $0 \leq i \leq s_n$,

$$L_n^{(\mathbf{s};\,j,i)}(u,q,z) = \begin{cases} 1 & \text{if } n = 0 \text{ or } j = i = 0, \text{ else} \\ L_n^{(\mathbf{s};\,j-1,s_n)}(u,q,z) & \text{if } i = 0, \text{ else} \\ L_n^{(\mathbf{s};\,j,i-1)}(u,q,z) + & \\ \quad u^{j+1} q^{js_n+i} z^{s_n-i} L_{n-1}^{(\mathbf{s};\,j,\lfloor is_{n-1}/s_n \rfloor)}(u,q,z) & \text{otherwise.} \end{cases}$$

Proof. The theorem is clearly true for $n = 0$ and for $j = i = 0$. Let (n, j, i) satisfy $n > 0$, $j, i \geq 0$, and $(j, i) \neq (0, 0)$. If $i = 0$, then $j > 0$ and $js_n + i = js_n = (j-1)s_n + s_n$, so the theorem is true. Assume, then, that $1 \leq i \leq s_n$. By definition, $\lambda \in \mathbf{L}_n^{(\mathbf{s};\,j,i)}$ if and only if either $\lambda \in \mathbf{L}_n^{(\mathbf{s};\,j,i-1)}$ or $\lambda \in \mathbf{L}_n^{(\mathbf{s})}$ and $\lambda_n = js_n + i$. But $(\lambda_1, \ldots, \lambda_{n-1}, js_n + i) \in \mathbf{L}_n^{(\mathbf{s})}$ if and only if $(\lambda_1, \ldots, \lambda_{n-1}) \in \mathbf{L}_{n-1}^{(\mathbf{s})}$ and $(js_n + i)/\lambda_{n-1} \geq s_n/s_{n-1}$. That is,

$$\lambda_{n-1} \;\leq\; \frac{s_{n-1}}{s_n}(js_n + i) \;=\; js_{n-1} + i\frac{s_{n-1}}{s_n}.$$

So, since λ_{n-1} is an integer,

$$\lambda_{n-1} \;\leq\; js_{n-1} + \left\lfloor i\frac{s_{n-1}}{s_n} \right\rfloor.$$

Note, since $1 \leq i \leq s_n$, $\lfloor is_{n-1}/s_n \rfloor \leq s_{n-1}$, so $(\lambda_1, \ldots, \lambda_{n-1}) \in \mathbf{L}_{n-1}^{(\mathbf{s};\,j,\lfloor is_{n-1}/s_n \rfloor)}$. Finally, adding $\lambda_n = js_n + i$ to the sequence $(\lambda_1, \ldots, \lambda_{n-1})$ multiplies its (u, q, z)-weight by $u^{\lceil \lambda_n/s_n \rceil} q^{|\lambda_n|} z^{s_n \lceil \lambda_n/s_n \rceil - \lambda_n} = u^{j+1} q^{js_n+i} z^{s_n-i}$. □

So, the refined Ehrhart polynomial (5) for the $\mathbf{s}$-lecture hall polytope, $\mathbf{P}_n^{(\mathbf{s})}$ is

$$\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) \;=\; L_n^{(\mathbf{s};\,t,0)}(u,q,z).$$

The following u-analog of a result from [8] can be proved by induction from Theorem 3.

Theorem 4. *For* $\mathbf{s} = (1, 2, \ldots, n)$, *and nonnegative integers* n, j, i *with* $0 \leq i \leq s_n = n$,

$$L_n^{(\mathbf{s};\,j,i)}(u,1,1) \;=\; [\,j+1\,]_u^{n-i}\,[\,j+2\,]_u^{i} \tag{8}$$

where $[\,j\,]_u = (1 - u^j)/(1 - u)$.

Corollary 1.

$$\mathbf{f}_n^{(1,2,\ldots,n)}(t; u, 1, 1) \;=\; [\,t+1\,]_u^n\,. \tag{9}$$

So by Corollary 1, the lecture hall polytope $\mathbf{P}_n^{(1,2,\ldots,n)}$ has the same u-Ehrhart polynomial as the unit cube $\mathbf{Q}_n$. This was proved bijectively in [5], motivating the current work.

2.4 Eulerian polynomials

Let $\mathfrak{S}_n$ denote the set of permutations of $\{1,\ldots,n\}$. For $\pi \in \mathfrak{S}_n$, a *descent* of π is a position i, $1 \le i < n$, such that $\pi_i > \pi_{i+1}$. Let $\mathrm{des}_{\mathrm{A}}\,\pi$ be the number of descents of π. The *major index* of π, $\mathrm{maj}_{\mathrm{A}}\,\pi$, is the sum of the descent positions. It was shown by MacMahon [21] that

$$\sum_{\pi \in \mathfrak{S}_n} u^{\mathrm{maj}_{\mathrm{A}}\,\pi} \;=\; \prod_{i=1}^{n} [\,i\,]_u\,.$$

We will use "u" for the statistics related to major index, reserving "q" for statistics related to the generating function of lecture hall partitions.

The distribution of $\mathrm{des}_{\mathrm{A}}$ over $\mathfrak{S}_n$ is $\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}_{\mathrm{A}}\,\pi} = \mathbf{E}_n(x)$ [22], where $\mathbf{E}_n(x)$ is the *Eulerian polynomial*, defined by

$$\sum_{t\ge 0}(t+1)^n x^t \;=\; \frac{\mathbf{E}_n(x)}{(1-x)^{n+1}}. \tag{10}$$

See [15] for an informative history of the Eulerian polynomials. A statistic on permutations is *Mahonian* if its distribution over $\mathfrak{S}_n$ is the same as $\mathrm{maj}_{\mathrm{A}}$ and *Eulerian* if it is the same as $\mathrm{des}_{\mathrm{A}}$. The joint distribution of $(\mathrm{des}_{\mathrm{A}}, \mathrm{maj}_{\mathrm{A}})$ over $\mathfrak{S}_n$ is given by Carlitz's refinement, $\mathbf{E}_n(x,u)$, of the Eulerian polynomials [6, 7]:

$$\sum_{t\ge 0}[\,t+1\,]_u^n\, x^t \;=\; \frac{\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}_{\mathrm{A}}\,\pi} u^{\mathrm{maj}_{\mathrm{A}}\,\pi}}{\prod_{i=0}^{n}(1-xu^i)} \;=\; \frac{\mathbf{E}_n(x,u)}{\prod_{i=0}^{n}(1-xu^i)}. \tag{11}$$

This is a special case of a result of MacMahon, Vol. 2, p. 211 in [20]. Equations (10) and (11) are interpreted combinatorially as relationships between integer points in the n-dimensional cube tQ_n and permutations in $\mathfrak{S}_n$.

2.5 Generalized inversion sequences

For $\pi \in \mathfrak{S}_n$, we define the *inversion sequence* of π, $e(\pi) = (e_1, e_2, \ldots e_n)$, by

$$e_i \;=\; \left|\{j \mid i > j \text{ and } \pi^{-1}(i) < \pi^{-1}(j)\}\right|.$$

The mapping $\pi \to e(\pi)$ is well-known to be a bijection between $\mathfrak{S}_n$ and $\{e \in \mathbb{Z}^n \mid 0 \le e_i < i\}$ [18]. The *inversion number* of π is

$$\mathrm{inv}_{\mathrm{A}}\,\pi \;=\; |e(\pi)|. \tag{12}$$

It is easy to check that $\sum_{\pi\in\mathfrak{S}_n} z^{\mathrm{inv}_{\mathrm{A}}\,\pi} = \prod_{i=1}^{n}[\,i\,]_z$, so that $\mathrm{inv}_{\mathrm{A}}$ is also Mahonian.

In this section we generalize the notion of an inversion sequence. Given a sequence $\mathbf{s} = \{s_i\}_{i\ge 1}$ of positive integers, define $\mathbf{I}_n^{(\mathbf{s})}$ by

$$\mathbf{I}_n^{(\mathbf{s})} \;=\; \{(e_1,\ldots,e_n) \in \mathbb{Z}^n \;\mid\; 0 \le e_i < s_i \text{ for } 1 \le i \le n\}\,.$$

The elements of $\mathbf{I}_n^{(\mathbf{s})}$ are the $\mathbf{s}$*-inversion sequences of length* n. For $e \in \mathbf{I}_n^{(\mathbf{s})}$, define the *ascent set* of e by

$$\mathrm{Asc}\,e \;=\; \{i \mid i = 0 \text{ and } e_1 > 0 \;\text{ or }\; 1 \le i < n \text{ and } \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}}\} \tag{13}$$

$\mathbf{s} = (1, 2, 3)$

$e \in \mathbf{I}_n^{(\mathbf{s})}$	$\lvert\cdots\rvert$	Asc	asc	amaj	lhp	Des	des	dmaj	dlhp
000	0	$\{\ \}$	0	0	0	$\{\ \}$	0	0	0
001	1	$\{2\}$	1	1	2	$\{\ \}$	0	0	1
002	2	$\{2\}$	1	1	1	$\{\ \}$	0	0	2
010	1	$\{1\}$	1	2	4	$\{2\}$	1	1	4
011	2	$\{1\}$	1	2	3	$\{2\}$	1	1	5
012	3	$\{1, 2\}$	2	3	5	$\{\ \}$	0	0	3

$\mathbf{s} = (3, 2, 1)$

$e \in \mathbf{I}_n^{(\mathbf{s})}$	$\lvert\cdots\rvert$	Asc	asc	amaj	lhp	Des	des	dmaj	dlhp
000	0	$\{\ \}$	0	0	0	$\{\ \}$	0	0	0
010	1	$\{1\}$	1	2	2	$\{2\}$	1	1	2
100	1	$\{0\}$	1	3	5	$\{1\}$	1	2	4
110	2	$\{0, 1\}$	2	5	7	$\{2\}$	1	1	3
200	2	$\{0\}$	1	3	4	$\{1\}$	1	2	5
210	3	$\{0\}$	1	3	3	$\{1, 2\}$	2	3	7

$\mathbf{s} = (2, 1, 2)$

$e \in \mathbf{I}_n^{(\mathbf{s})}$	$\lvert\cdots\rvert$	Asc	asc	amaj	lhp	Des	des	dmaj	dlhp
000	0	$\{\ \}$	0	0	0	$\{\ \}$	0	0	0
001	1	$\{2\}$	1	1	1	$\{\ \}$	0	0	1
100	1	$\{0\}$	1	3	4	$\{1\}$	1	2	4
101	2	$\{0, 2\}$	2	4	5	$\{1\}$	1	2	5

Table 1: Statistics on inversion sequences $\mathbf{I}_n^{(\mathbf{s})}$ for various sequences $\mathbf{s}$.

and let $\operatorname{asc} e = |\operatorname{Asc} e|$. Define the set of *descents* of $e \in \mathbf{I}_n^{(\mathbf{s})}$ by

$$\operatorname{Des} e \;=\; \left\{ i : 1 \leq i < n \;\middle|\; \frac{e_i}{s_i} > \frac{e_{i+1}}{s_{i+1}} \right\}$$

and $\operatorname{des} e = |\operatorname{Des} e|$. Inversion sequence statistics are illustrated in Table 1. From the examples in Table 1, note that asc and des are not necessarily equidistributed over $\mathbf{s}$-inversion sequences.

For $e \in \mathbf{I}_n^{(\mathbf{s})}$, define the statistics amaj and dmaj by

$$\operatorname{amaj} e \;=\; \sum_{i \in \operatorname{Asc} e} (n-i),$$
$$\operatorname{dmaj} e \;=\; \sum_{i \in \operatorname{Des} e} (n-i).$$

Lemma 1. *For* $\mathbf{s} = (1, 2, \ldots, n)$,

$$\sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}_{\mathrm{A}} \pi} u^{\operatorname{maj}_{\mathrm{A}} \pi} z^{\operatorname{inv}_{\mathrm{A}} \pi} \;=\; \sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e} u^{\operatorname{amaj} e} z^{|e|}. \tag{14}$$

Proof. For $\pi \in \mathfrak{S}_n$, $\operatorname{inv}_{\mathrm{A}} \pi = \operatorname{inv}_{\mathrm{A}} \pi^{-1}$. Also, i is a descent of π if and only if $i \in \operatorname{Asc} e(\pi^{-1})$. So the mapping $\pi \to e(\pi^{-1})$ sends $(\operatorname{des}_{\mathrm{A}}, \operatorname{inv}_{\mathrm{A}})$ to $(\operatorname{asc}, |e|)$.

The mapping $\pi \to \sigma$ defined by $\sigma_i = n+1-\pi_{n+1-i}$ also preserves $\operatorname{inv}_{\mathrm{A}}$. Further, i is a descent of π if and only if $n-i$ is a descent of σ. So the composition: $\pi \to \sigma \to e(\sigma^{-1})$ sends $(\operatorname{des}_{\mathrm{A}}, \operatorname{maj}_{\mathrm{A}}, \operatorname{inv}_{\mathrm{A}})$ to $(\operatorname{asc}, \operatorname{amaj}, |e|)$. □

In view of (9), Lemma 1, and (11), when $\mathbf{s} = (1, 2, \ldots, n)$, we can re-interpret Carlitz's identity (11) as a relationship between $\mathbf{s}$-*lecture hall partitions* and $\mathbf{s}$-*inversion sequences*:

$$\sum_{t \geq 0} [\,t+1]^n \, x^t \;=\; \sum_{t \geq 0} \mathbf{f}_n^{(\mathbf{s})}(t, u, 1, 1) x^t \;=\; \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e} u^{\operatorname{amaj} e}}{\prod_{i=0}^{n} (1 - x u^i)}. \tag{15}$$

Finally, we introduce the "lecture hall" statistics, lhp and dlhp on $\mathbf{I}_n^{(\mathbf{s})}$ (see Table 1):

$$\operatorname{lhp} e \;=\; -|e| + \sum_{i \in \operatorname{Asc} e} (s_{i+1} + \cdots + s_n),$$
$$\operatorname{dlhp} e \;=\; |e| + \sum_{i \in \operatorname{Des} e} (s_{i+1} + \cdots + s_n).$$

2.6 Barred inversion sequences

In [17], Gessel and Stanley define a *barred permutation* of $\mathfrak{S}_n$ to be a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ in which one or more vertical bars are inserted before and/or after elements π_i, with the stipulation that if i is a descent of π, there is at least one bar between π_i and π_{i+1}. Counting barred permutations in two ways gives an elegant proof of (10). First, fix the number of bars, t and sum over $t \geq 0$. Then fix the permutation π, and sum over all π.

| | 0 0 0	0 | | 0 0	0 0 | | 0	0 0 0 | |	| 0 | 0 0
| 0 0 | 0	| 0 0 0 |	0 | 0 | 0	0 | 0 0 |	0 0 | 0 |
| 0 0 | 1	0 | 0 | 1	0 0 | | 1	0 0 | 1 |	
| | 1 0 0	| 1 | 0 0	| 1 0 | 0	| 1 0 0 |	
| 1 0 | 1				

Figure 1: Barred inversion sequences of $\mathbf{I}_3^{(2,1,2)}$ with exactly 2 bars.

We will adapt this tool for inversion sequences. Define a *barred inversion sequence* of $\mathbf{I}_n^{(\mathbf{s})}$ to be a sequence $e \in \mathbf{I}_n^{(\mathbf{s})}$ in which one or more vertical bars are inserted before and/or after elements e_i, with the stipulation that if i is an *ascent* of e, there is at least one bar in position i, the space immediately preceding e_{i+1}. Using the example $\mathbf{s} = (2, 1, 2)$ from Table 1, for $\mathbf{I}_3^{(2,1,2)}$ there are 19 barred inversion sequences with exactly two bars (see Figure 1).

3 Main results

Recall that

$$\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) \;=\; \sum_{\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap Z^n} u^{\,|\lceil \lambda \rceil|}\, q^{\,|\lambda|}\, z^{\,|\,\epsilon^+(\lambda)\,|},$$

where

$$t\mathbf{P}_n^{(\mathbf{s})} \;=\; \left\{ \lambda \in \mathbb{R}^n \;\middle|\; 0 \le \frac{\lambda_1}{s_1} \le \frac{\lambda_2}{s_2} \le \cdots \le \frac{\lambda_n}{s_n} \le t \right\}.$$

We can now state and prove our main result relating the Ehrhart series of $\mathbf{s}$-lecture hall polytopes to statistics on $\mathbf{s}$-inversion sequences.

Theorem 5. *Let* $\mathbf{s}$ *be any sequence of positive integers. For integer* $n \ge 0$*,*

$$\sum_{t \ge 0} \mathbf{f}_n^{(\mathbf{s})}(t; u, q, z)\, x^t \;=\; \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}\, u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|}}{\prod_{i=0}^{n}(1 - x u^{n-i} q^{s_{i+1}+\cdots+s_n})}. \tag{16}$$

Proof. Let e be a barred $\mathbf{s}$-inversion sequence with d_i bars in position i for $0 \le i \le n$. Assign to e the weight

$$w(e) \;=\; \left(\frac{z}{q}\right)^{|\,e\,|} \prod_{i=0}^{n} (x u^{n-i} q^{s_{i+1}+\cdots+s_n})^{d_i}. \tag{17}$$

For example, if $\mathbf{s} = (2, 1, 2)$, the barred $\mathbf{s}$-inversion sequence $|1||0|1|$ has the weight:

$$w(|1||0|1|) \;=\; (z/q)^2 (xu^3q^5)(xu^2q^3)^2(xuq^2)(x) \;=\; x^5 z^2 u^8 q^{11}.$$

We show that both sides of (16) count barred inversion sequences by counting barred $\mathbf{s}$-inversion sequences in two ways. First, fix the inversion sequence, e, and sum over all $e \in \mathbf{I}_n^{(\mathbf{s})}$. Second, fix the number of bars, t, and sum over $t \geq 0$.

The first way is easy: a barred inversion sequence $e \in I_n^{(\mathbf{s})}$ must have at least one bar following each ascent position. This minimally contributes $x^{\operatorname{asc} e}\, u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\, e\,|}$ to $w(e)$. But in each position i of e, any number $j \geq 0$ of additional bars may be placed, contributing a term in the series below as an additional factor to $w(e)$:

$$\sum_{j=0}^{\infty} (xu^{n-i} q^{s_{i+1}+\cdots+s_n})^j \;=\; \frac{1}{1 - xu^{n-i} q^{s_{i+1}+\cdots+s_n}}.$$

So, the right-hand side of (16) counts all barred inversion sequences of $\mathbf{I}_n^{(\mathbf{s})}$, with their corresponding weight.

For the left-hand side of (16), we describe a weight-preserving bijection from $\mathbf{s}$-lecture hall partitions in $t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ to barred $\mathbf{s}$-inversion sequences with t bars. If $\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$, let

$$b \;=\; \lceil \lambda \rceil \;=\; \left(\left\lceil \frac{\lambda_1}{s_1} \right\rceil, \left\lceil \frac{\lambda_2}{s_2} \right\rceil, \ldots, \left\lceil \frac{\lambda_n}{s_n} \right\rceil \right).$$

Note that $b_n \leq t$. Let $e = \epsilon^+(\lambda) = (e_1, e_2, \ldots, e_n)$, so

$$e_i \;=\; s_i b_i - \lambda_i.$$

Then clearly $e \in \mathbf{I}_n^{(\mathbf{s})}$. We "bar" e with t bars by placing b_1 bars before e_1; $b_i - b_{i-1}$ bars before e_i for $2 \leq i \leq n$; and $t - b_n$ bars after e_n. Since λ was an $\mathbf{s}$-lecture hall partition, if $1 \leq i < n$ and there is no bar after e_i, then $b_i = b_{i+1}$ and therefore

$$\frac{e_i}{s_i} \;=\; b_i - \frac{\lambda_i}{s_i} \;\geq\; b_i - \frac{\lambda_{i+1}}{s_{i+1}} \;=\; b_{i+1} - \frac{\lambda_{i+1}}{s_{i+1}} \;=\; \frac{e_{i+1}}{s_{i+1}}.$$

Thus $i \notin \operatorname{Asc}(e)$. If there is no bar before e_1, then $b_1 = 0$, so $\lambda_1 = 0$ and thus $e_1 = 0$ and so in this case also, $0 \notin \operatorname{Asc} e$. Thus e is a barred inversion sequence with t bars. For example, if $\mathbf{s} = (2,1,2)$ and $t = 5$, let $\lambda = (1,3,7) \in t\mathbf{P}_3^{(\mathbf{s})} \cap \mathbb{Z}^3$. Then $b = \lceil \lambda \rceil = (1,3,4)$ and $\epsilon^+(\lambda) = (1,0,1)$, so λ maps to the barred $\mathbf{s}$-inversion sequence $|1||0|1|$.

Before specifying the inverse, note that this mapping preserves weights, as follows. For fixed t, the weight associated with $\lambda \in t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ is

$$w(\lambda) \;=\; u^{\,|\lceil \lambda \rceil|}\, q^{\,|\lambda|}\, z^{\,|\,\epsilon^+(\lambda)\,|}\, x^t,$$

whereas, from (17), the weight of the barred inversion sequence $e = \epsilon^+(\lambda)$, derived from λ, is

$$w(e) \;=\; \left(\frac{z}{q}\right)^{|\,e\,|} \prod_{i=0}^{n} (xu^{n-i} q^{s_{i+1}+\cdots+s_n})^{d_i},$$

where $d_0 = b_1$, $d_n = t - b_n$, and for $1 \leq i \leq n-1$, $d_i = b_{i+1} - b_i$. Comparing the exponents of z, x, u, and q in $w(\lambda)$ and $w(e)$ we find that they agree: $|e| = |\epsilon^+(\lambda)|$ and

$$\begin{aligned} \sum_{i=0}^{n} d_i &= b_1 + (t - b_n) + \sum_{i=1}^{n-1}(b_{i+1} - b_i) = t; \\ \sum_{i=0}^{n}(n-i)d_i &= nb_1 + \sum_{i=1}^{n-1}(n-i)(b_{i+1} - b_i) = |b|; \\ -|e| + \sum_{i=0}^{n}(s_{i+1} + \cdots + s_n)d_i &= -|e| + \sum_{i=1}^{n} s_i b_i = |\lambda|. \end{aligned}$$

Finally, to prove that the mapping from $t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ to barred **s**-inversion sequences with t bars is a bijection, we specify its inverse. Let e be a barred **s**-inversion sequence with t bars. For $1 \leq i \leq n$, let b_i be the total number of bars preceding e_i in any position. Then $b_1 \leq b_2 \leq \ldots \leq b_n \leq t$. Define $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_n)$ by

$$\lambda_i = s_i b_i - e_i.$$

We show $\lambda \in t\mathbf{P}_n^{(\mathbf{s})}$. Then $\lambda_n/s_n = (s_n b_n - e_n)/s_n \leq t$, since $e_n \geq 0$. First note that $\lambda_i \geq 0$, since $e_i < s_i$ and if $b_i = 0$, then, by definition of barred inversion sequence, no position $j < i$ is an ascent of e, so $e_1 = e_2 = \ldots = e_i = 0$. Secondly, we show that λ is an **s**-lecture hall partition. Since e is an **s**-inversion sequence, $\frac{e_j}{s_j} < 1$ for all j. Now, if $i \in \operatorname{Asc} e$, there is at least one bar between e_i and e_{i+1}, so $b_i < b_{i+1}$ and

$$\frac{\lambda_i}{s_i} = \frac{s_i b_i - e_i}{s_i} = b_i - \frac{e_i}{s_i} \leq b_i \leq b_{i+1} - 1 < b_{i+1} - \frac{e_{i+1}}{s_{i+1}} = \frac{\lambda_{i+1}}{s_{i+1}}.$$

On the other hand, if $i \notin \operatorname{Asc} e$, then

$$\frac{e_i}{s_i} \geq \frac{e_{i+1}}{s_{i+1}},$$

so

$$\frac{\lambda_i}{s_i} = \frac{s_i b_i - e_i}{s_i} = b_i - \frac{e_i}{s_i} \leq b_{i+1} - \frac{e_{i+1}}{s_{i+1}} = \frac{\lambda_{i+1}}{s_{i+1}}.$$

This completes the proof. □

The generating function for the set $\mathbf{L}_n^{(\mathbf{s})}$ of **s**-lecture hall partitions can now be expressed in terms of statistics on **s**-inversion sequences. Recall that

$$\mathbf{L}_n^{(\mathbf{s})} = \{\lambda \in \mathbb{Z}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n}\}.$$

Theorem 6. The **s**-Lecture Hall Theorem: *For any sequence* **s** *of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} u^{|\lceil \lambda \rceil|} q^{|\lambda|} z^{|\epsilon^+(\lambda)|} x^{\lceil \lambda_n/s_n \rceil} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e} u^{\operatorname{amaj} e} q^{\operatorname{lhp} e} z^{|e|}}{\prod_{i=0}^{n-1}(1 - xu^{n-i}q^{s_{i+1}+\cdots+s_n})}. \tag{18}$$

Proof. For $\lambda \in \mathbf{L}_n^{(\mathbf{s})}$, $\lambda \in t\mathbf{P}_n^{(\mathbf{s})}$ if and only if $\lambda_n/s_n \le t$. So for $t > 0$, the generating function for the elements of $\mathbf{L}_n^{(\mathbf{s})}$ with $\lceil \lambda_n/s_n \rceil = t$ is

$$\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) - \mathbf{f}_n^{(\mathbf{s})}(t-1; u, q, z).$$

Thus,

$$\begin{aligned}
\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|}\, u^{|\lceil \lambda \rceil|}\, z^{|\,\epsilon^+(\lambda)\,|}\, x^{\lceil \lambda_n/s_n \rceil} &= 1 + \sum_{t \ge 1} (\mathbf{f}_n^{(\mathbf{s})}(t; u, q, z) - \mathbf{f}_n^{(\mathbf{s})}(t-1; u, q, z))\, x^t \\
&= \sum_{t \ge 0} \mathbf{f}_n^{(\mathbf{s})}(t; u, q, z)\, x^t - x \sum_{t \ge 1} \mathbf{f}_n^{(\mathbf{s})}(t-1; u, q, z))\, x^{t-1} \\
&= (1-x) \sum_{t \ge 0} \mathbf{f}_n^{(\mathbf{s})}(t; u, q, z)\, x^t \\
&= \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}\, u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|}}{\prod_{i=0}^{n-1} (1 - x u^{n-i} q^{s_{i+1} + \cdots + s_n})},
\end{aligned}$$

where the last equality is Theorem 5. □

So, we can now interpret statistics on lecture hall partitions in terms of asc , amaj and lhp on inversion sequences.

Corollary 2. *For any sequence,* $\mathbf{s}$*, of positive integers, and for integer* $n \ge 0$*,*

$$\text{(a)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{lhp} e}}{\prod_{i=1}^{n} (1 - q^{s_i + \cdots + s_n})}$$

$$\text{(b)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lceil \lambda_n/s_n \rceil} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e}}{(1-x)^n}$$

$$\text{(c)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} u^{|\lceil \lambda \rceil|} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{amaj} e}}{\prod_{i=1}^{n} (1 - u^i)}.$$

Note that although we came to Theorem 6 via the lecture hall polytope, in retrospect, everything can be done directly on lecture hall partitions using the barred inversion sequences. And instead of using $\lceil \lambda \rceil$, and ascent statistics, everything works equally well with $\lfloor \lambda \rfloor$ and descent statistics. This gives rise to the following descent version of the $\mathbf{s}$-Lecture Hall Theorem. Details, which are very similar, are omitted.

Theorem 7. The $\mathbf{s}$-Lecture Hall Theorem (descent version): *For any sequence* $\mathbf{s}$ *of positive integers,*

$$\sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} u^{|\lfloor \lambda \rfloor|}\, q^{|\lambda|}\, z^{|\,\epsilon^-(\lambda)\,|}\, x^{\lfloor \lambda_n/s_n \rfloor} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{des} e}\, u^{\operatorname{dmaj} e}\, q^{\operatorname{dlhp} e}\, z^{|\,e\,|}}{\prod_{i=0}^{n-1} (1 - x u^{n-i} q^{s_{i+1} + \cdots + s_n})}. \tag{19}$$

The statistics des, dmaj, and dlhp on inversion sequences can now also be interpreted in terms of corresponding statistics on lecture hall partitions.

Corollary 3. *For any sequence,* $\mathbf{s}$*, of positive integers, and for integer* $n \geq 0$*,*

$$\text{(a)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{dlhp} e}}{\prod_{i=1}^{n}(1 - q^{s_i + \cdots + s_n})}$$

$$\text{(b)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} x^{\lfloor \lambda_n / s_n \rfloor} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{des} e}}{(1 - x)^n}$$

$$\text{(c)} \quad \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} u^{|\lfloor \lambda \rfloor|} = \frac{\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} u^{\operatorname{dmaj} e}}{\prod_{i=1}^{n}(1 - u^i)}.$$

We get the following not-so-obvious equidistribution results.

Corollary 4. *For any sequence,* $\mathbf{s}$*, of positive integers,*

$$\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{lhp} e} = \sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} q^{\operatorname{dlhp} e}.$$

If, in addition $s_n = 1$*, then*

$$\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e} q^{\operatorname{lhp} e} = \sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{des} e} q^{\operatorname{dlhp} e}.$$

Proof. The first equality comes from the first identities in Corollaries 2 and 3. For the second, set $u = z = 1$ in Theorems 6 and 7 and observe that if $s_n = 1$, then

$$\lceil \lambda_n / s_n \rceil = \lambda_n = \lfloor \lambda_n / s_n \rfloor. \qquad \square$$

4 Examples and Applications

All unattributed theorems, lemmas, and corollaries in this section are, to the best of our knowledge, new to this paper.

4.1 The q-binomial theorem

To illustrate how Theorem 5 can be applied, we show first that it can be viewed as a generalization of Newton's q-binomial theorem: When $\mathbf{s} = (1, 1, \ldots, 1)$, the $\mathbf{s}$-inversion sequence $(0, 0, \ldots, 0)$ is the only element of $\mathbf{I}_n^{(\mathbf{s})}$. The $\mathbf{s}$-lecture hall polytope, $\mathbf{P}_n^{(1,1,\ldots,1)}$ consists of $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq 1$. Therefore, the Ehrhart polynomial of $\mathbf{P}_n^{(1,1,\ldots,1)}$ counts ordinary partitions with at most n positive parts and with no part greater than t. So, $\mathbf{f}_n^{(\mathbf{s})}(t;1,q,1)$ is given by the q-binomial coefficient, $\begin{bmatrix} n+t \\ t \end{bmatrix}_q$. Then (16) becomes

$$\sum_{t\geq 0} \begin{bmatrix} n+t \\ t \end{bmatrix}_q x^t \;=\; \prod_{i=0}^{n} \frac{1}{1-xq^i}. \tag{20}$$

4.2 The s-Eulerian numbers

The theory enables us to prove new results generalizing the Eulerian numbers.

First note that Theorem 5 can be viewed as a generalization of Carlitz's generating function (11) for the u-Eulerian polynomials. Set $q = z = 1$ and $\mathbf{s} = (1, 2, \ldots, n)$ in (16) to get

$$\sum_{t\geq 0} \mathbf{f}_n^{(1,2,\ldots,n)}(t;u,1,1)\, x^t \;=\; \frac{\sum_{e\in \mathbf{I}_n^{(1,2,\ldots,n)}} x^{\operatorname{asc} e}\, u^{\operatorname{amaj} e}}{\prod_{i=0}^{n}(1-xu^i)}.$$

By (9), $\mathbf{f}_n^{(1,2,\ldots,n)}(t;u,1,1) = [t+1]_u^n$. By Lemma 1, (asc, amaj) over $\mathbf{I}_n^{(1,2,\ldots,n)}$ has the same joint distribution as $(\operatorname{des}_{\mathrm{A}}, \operatorname{maj}_{\mathrm{A}})$ over $\mathfrak{S}_n$. This gives (11).

With this motivation, for a positive integer sequence $\mathbf{s}$, and $n \geq 0$, define the n-th $\mathbf{s}$-*Eulerian polynomial* by

$$\mathbf{E}_n^{(\mathbf{s})}(x) \;=\; \sum_{e\in \mathbf{I}_n^{(\mathbf{s})}} x^{\operatorname{asc} e} \;=\; \sum_{k=0}^{n} a_{n,k}^{(\mathbf{s})} x^k.$$

Then $a_{n,k}^{(\mathbf{s})}$ is the number of $\mathbf{s}$-inversion sequences of length n with k ascents. The numbers $a_{n,k}^{(\mathbf{s})}$ are the $\mathbf{s}$-*Eulerian numbers.* When $\mathbf{s} = (1, 2, 3, \ldots)$, $\mathbf{E}_n^{(\mathbf{s})}(x) = \mathbf{E}_n(x)$ of (10).

For $t \geq 0$, let $b_{n,t}^{(\mathbf{s})}$ be the number of $\mathbf{s}$-lecture hall partitions $\lambda \in \mathbf{L}_n^{(\mathbf{s})}$ with $\lambda_n/s_n \leq t$ (so $b_{n,t}^{(\mathbf{s})} = \mathbf{f}_n^{(\mathbf{s})}(t)$.) The $\mathbf{s}$-Eulerian numbers are related to the $b_{n,t}^{(\mathbf{s})}$ as follows.

Theorem 8.

$$\sum_{t\geq 0} b_{n,t}^{(\mathbf{s})} x^t \;=\; \frac{\sum_{k=0}^{n} a_{n,k}^{(\mathbf{s})} x^k}{(1-x)^{n+1}};$$

$$a_{n,k}^{(\mathbf{s})} \;=\; \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} b_{n,k-i}^{(\mathbf{s})}.$$

Proof. The first identity is Theorem 5 with $u = q = z = 1$. The second identity follows from the first, using the binomial expansion of $(1-x)^{n+1}$ and then equating coefficients of x^k. □

In fact, we get the following $\mathbf{s}$-generalization of Carlitz's polynomial analog of the

Eulerian numbers. Define polynomials $a^{(\mathbf{s})}_{n,k}(u)$ and $b^{(\mathbf{s})}_{n,t}(u)$ by

$$\begin{aligned} a^{(\mathbf{s})}_{n,k}(u) &= \sum_{e \in \mathbf{I}^{(\mathbf{s})}_n;\ \operatorname{asc} e = k} u^{\operatorname{amaj} e}; \\ b^{(\mathbf{s})}_{n,t}(u) &= \sum_{\lambda \in \mathbf{L}^{(\mathbf{s})}_n;\ \lambda_n/s_n \le t} u^{|\lceil \lambda \rceil|}. \end{aligned}$$

Theorem 9. *For $n \geq k \geq 0$ and positive integer sequence* $\mathbf{s}$,

$$\sum_{t \geq 0} b^{(\mathbf{s})}_{n,t}(u) x^t = \frac{\sum_{k=0}^{n} a^{(\mathbf{s})}_{n,k}(u) x^k}{\prod_{i=0}^{n}(1 - xu^i)};$$

$$a^{(\mathbf{s})}_{n,k}(u) = \sum_{i=0}^{k} (-1)^i u^{i(i+1)/2} \begin{bmatrix} n+1 \\ i \end{bmatrix}_u b^{(\mathbf{s})}_{n,k-i}(u).$$

4.3 Eulerian polynomials and the lhp statistic

The lecture hall statistic gives rise to a new and interesting q-analog of the familiar Eulerian polynomials. Using Theorem 6, we will apply lecture hall theory to prove that distribution of this statistic has a surprisingly simple product form.

Setting $u = z = 1$ in Theorem 6, with $\mathbf{s} = (1, 2, \ldots, n)$, provides an interpretation of a new q-analog of the Eulerian polynomials:

$$\sum_{\lambda \in \mathbf{L}^{(1,2,\ldots,n)}_n} q^{|\lambda|} x^{\lceil \lambda_n / n \rceil} = \frac{\sum_{e \in \mathbf{I}^{(1,2,\ldots,n)}_n} x^{\operatorname{asc} e} q^{\operatorname{lhp} e}}{\prod_{i=1}^{n}(1 - xq^{(i+\cdots+n)})}. \tag{21}$$

Setting $x = 1$ in (21) and applying the original lecture hall partitions generating function (1) gives the distribution of lhp over $(1, 2, \ldots, n)$-inversion sequences which, by Corollary 4, has the same distribution as dlhp for any sequence $\mathbf{s}$:

Corollary 5.

$$\sum_{e \in \mathbf{I}^{(1,2,\ldots,n)}_n} q^{\operatorname{lhp} e} = \prod_{k=1}^{n} [\, k \,]_{q^{2(n-k)+1}} = \sum_{e \in \mathbf{I}^{(1,2,\ldots,n)}_n} q^{\operatorname{dlhp} e}.$$

Proof. From (1) and Corollary 2(a),

$$\begin{aligned} \sum_{e \in \mathbf{I}^{(1,2,\ldots,n)}_n} q^{\operatorname{lhp} e} &= \prod_{k=1}^{n} \frac{1 - q^{k+\cdots+n}}{1 - q^{2k-1}} \\ &= \prod_{k=1}^{\lfloor n/2 \rfloor} \frac{1 - q^{(2k-1)(n-k+1)}}{1 - q^{2k-1}} \prod_{k=1}^{\lceil n/2 \rceil} \frac{1 - q^{k(2(n-k)+1)}}{1 - q^{2(n-k)+1}} \\ &= \prod_{k=1}^{n} [\, k \,]_{q^{2(n-k)+1}} . \end{aligned}$$

□

4.4 Mahonian statistics and lecture hall statistics

In this section, we combine Theorems 6 and 7 with the refined theorems for lecture hall partitions (2) and anti-lecture hall compositions (3) to derive the joint distribution of familiar Mahonian statistics and the new lecture hall statistics. This leads to some unusual generating functions for $n!$.

Corollary 6.

$$\sum_{e\in\mathbf{I}_n^{(n,n-1,\ldots,1)}} u^{\mathrm{dmaj}\, e}\, q^{\mathrm{dlhp}\, e} \;=\; \prod_{k=1}^{\lceil n/2\rceil} [2k-1]_{uq^k} \prod_{k=1}^{\lfloor n/2\rfloor} ([2]_{uq^{n-k+1}}\, [k]_{u^2q^{2k+1}}).$$

Proof. Apply Theorem 7 with $x = z = 1$ and $\mathbf{s} = (n, n-1, \ldots, 1)$. Combine with (3) to get

$$\sum_{e\in\mathbf{I}_n^{(n,n-1,\ldots,1)}} u^{\mathrm{dmaj}\, e} q^{\mathrm{dlhp}\, e} \;=\; \prod_{k=1}^{n} \frac{(1+uq^k)(1-u^k q^{k(k+1)/2})}{1-u^2q^{k+1}} \;\triangleq\; \alpha_n(q,u). \qquad (22)$$

Observe that $\alpha_n(q,u)$ satisfies the recurrence

$$\frac{\alpha_n(q,u)}{\alpha_{n-1}(q,u)} \;=\; \frac{(1+uq^n)(1-u^n q^{n(n+1)/2})}{1-u^2q^{n+1}},$$

with initial condition $\alpha_1(q,u) = 1$. It can be checked that the right-hand-side of the corollary satisfies the same recurrence and initial conditions. □

Corollary 7.

$$\sum_{e\in\mathbf{I}_n^{(1,2,\ldots,n)}} u^{\mathrm{amaj}\, e} q^{\mathrm{lhp}\, e} \;=\; \prod_{k=1}^{\lceil n/2\rceil} [2k-1]_{uq^{n+1-k}} \prod_{k=1}^{\lfloor n/2\rfloor} ([2]_{uq^k}\, [k]_{u^2q^{2(n-k)+1}}).$$

Proof. Apply Theorem 6 with $x = z = 1$ and $\mathbf{s} = (1, 2, \ldots, n)$. Combine with (2) to get

$$\sum_{e\in\mathbf{I}_n^{(1,2,\ldots,n)}} q^{\mathrm{lhp}\, e} u^{\mathrm{amaj}\, e} \;=\; \prod_{k=1}^{n} \frac{(1+uq^k)(1-u^{n+1-k} q^{k+\ldots n})}{1-u^2q^{n+k}} \;\triangleq\; \beta_n(q,u).$$

Observe that, using α from (22),

$$\beta_n(q,u) \;=\; \alpha_n(1/q, uq^{n+1}).$$

Thus setting $q = 1/q$ and then $u = uq^{n+1}$ in the right-hand-side of the equation on the Corollary gives the result. □

4.5 k-ary words and lecture hall statistics

When $\mathbf{s} = (k, k, \ldots, k)$, $\mathbf{I}_n^{(\mathbf{s})} = [k]^n$, where $[k] = \{0, 1, \ldots, k-1\}$. The statistics on $\mathbf{s}$-inversion sequences become statistics on words in $[k]^n$ and Theorems 5 and 6 apply. We give just a few examples of new results that follow.

Let P_n be the set of partitions into n nonnegative parts. Apply Theorem 5, setting $u = z = 1$ and $\mathbf{s} = (k, k, \ldots, k)$ in (16). Then the elements of $t\mathbf{P}_n^{(\mathbf{s})} \cap \mathbb{Z}^n$ are the elements of P_n with largest part at most kt. Thus $\mathbf{f}_n(t; 1, q, 1) = \left[{n+kt \atop n}\right]_q$. This gives the following analog of the q-Eulerian polynomials.

Corollary 8. *For* $k \geq 1$*,*

$$\sum_{t \geq 0} \begin{bmatrix} n+kt \\ n \end{bmatrix}_q x^t \;=\; \frac{\sum_{e \in \mathbf{I}_n^{(k,k,\ldots,k)}} x^{\operatorname{asc} e}\, q^{\operatorname{lhp} e}}{\prod_{i=0}^{n}(1 - xq^{ki})} \;=\; \frac{\sum_{w \in [k]^n} x^{\operatorname{asc} w}\, q^{\operatorname{lhp} w}}{\prod_{i=0}^{n}(1 - xq^{ki})}.$$

If we instead apply Theorem 6, setting $u = z = 1$ and $\mathbf{s} = (k, k, \ldots, k)$ in (18), and observe that $P_n = \mathbf{L}_n^{(k,k,\ldots,k)}$, we arrive at the following:

Corollary 9. *For* $k \geq 1$*,*

$$\sum_{\lambda \in P_n} q^{|\lambda|}\, x^{\lceil \lambda_n / k \rceil} \;=\; \frac{\sum_{w \in [k]^n} x^{\operatorname{asc} w}\, q^{\operatorname{lhp} w}}{\prod_{i=1}^{n}(1 - xq^{ki})}.$$

To get the distribution of lhp over $[k]^n$, set $x = 1$ in Corollary 9 and use the fact that $\sum_{\lambda \in P_n} q^{|\lambda|} = \prod_{i=1}^{n} 1/(1 - q^i)$.

Theorem 10. *For* $k \geq 1$*,*

$$\sum_{w \in [k]^n} q^{\operatorname{lhp} w} \;=\; \prod_{i=1}^{n} [\, k \,]_{q^i}$$

It is straightforward to check that $\sum_{i=1}^{n} i w_i$ has the same distribution as $\operatorname{lhp} w$ over $[k]^n$, but we do not have a combinatorial explanation of this fact.

4.6 Signed Eulerian polynomials

Let $\mathbf{E}_n(x, u)$ denote the usual u-Eulerian polynomials. The *signed u-Eulerian polynomials* of Wachs [29] are defined as

$$B_n(x, u) \;=\; \sum_{\pi \in \mathfrak{S}_n} (-1)^{\operatorname{inv_A} \pi} x^{\operatorname{des} \pi} u^{\operatorname{maj_A} \pi}.$$

Wachs [29] proved the (even case of the) following theorem which was a generalization of a conjecture of Loday [19] when $u = 1$ and of a generating function of Gessel and Simion when $x = 1$. See also reference [16]. The odd case here is new.

Theorem 11.

$$\begin{aligned} B_{2n}(x, u) &= (xu; u^2)_n \mathbf{E}_n(x, u^2); \qquad \text{(Wachs [29])} \\ B_{2n+1}(x, u) &= ((xu; u^2)_{n+1} \mathbf{E}_n(x, u^2) - u(x; u^2)_{n+1} \mathbf{E}_n(ux, u^2))/(1 - u). \end{aligned}$$

We show how to use our main results to derive Theorem 11. First use Lemma 1 and then Theorem 5 with $q = 1$ and $z = -1$ to get:

$$B_n(x,u) = \sum_{e \in I_n^{(\mathbf{s})}} (-1)^{|e|} x^{\operatorname{asc} e} u^{\operatorname{amaj} e} \tag{23}$$

$$= (x;u)_{n+1} \sum_{t\geq 0} \mathbf{f}_n^{(\mathbf{s})}(t;u,1,-1)x^t. \tag{24}$$

Then use the recurrence of Theorem 3 to find $\mathbf{f}_n^{(\mathbf{s})}(t;u,1,-1)$. The following can be proved by induction from Theorem 3.

Lemma 2.

$$L_n^{(\mathbf{s};\,j,i)}(u,1,-1) = [\,j+1\,]_{u^2}^{\lfloor (n-i)/2\rfloor} [\,j+2\,]_{u^2}^{\lfloor i/2\rfloor} \alpha(n,i,j),$$

where, if n is odd,

$$\alpha(n,i,j) = \begin{cases} [\,j+1\,]_u & \textit{if } i \textit{ is even,} \\ [\,j+2\,]_u & \textit{if } i \textit{ is odd;} \end{cases}$$

and, if n is even,

$$\alpha(n,i,j) = \begin{cases} 1 & \textit{if } i \textit{ is even,} \\ (1-u^{j+1})(1-u^{j+2})/(1-u^2) & \textit{if } i \textit{ is odd.} \end{cases}$$

Recall that $\mathbf{f}_n^{(\mathbf{s})}(t;u,1,-1) = L_n^{(\mathbf{s};\,t,0)}(u,1,-1)$. Substitution into Lemma 2 gives:

Corollary 10.

$$f_{2n}^{(\mathbf{s})}(t;u,1,-1) = [\,t+1\,]_{u^2}^n\,;$$
$$f_{2n+1}^{(\mathbf{s})}(t;u,1,-1) = [\,t+1\,]_{u^2}^n\,[\,t+1\,]_u\,.$$

Theorem 11 follows now by substitution into (24). For even dimension we get

$$\begin{aligned} B_{2n}(x,u) &= (x;u)_{2n+1} \sum_{t\geq 0} \mathbf{f}_{2n}^{(\mathbf{s})}(t;u,1,-1)x^t \\ &= (x;u)_{2n+1} \sum_{t\geq 0} [\,t+1\,]_{u^2}^n\, x^t \\ &= (x;u)_{2n+1}(x;u^2)_n \mathbf{E}_n(x,u^2) \;=\; (xu;u^2)_n \mathbf{E}_n(x,u^2). \end{aligned}$$

For odd dimension we get

$$\begin{aligned} B_{2n+1}(x,u) &= (x;u)_{2n+2} \sum_{t\geq 0} \mathbf{f}_{2n+1}^{(\mathbf{s})}(t;u,1,-1)x^t \\ &= (x;u)_{2n+2} \sum_{t\geq 0} [\,t+1\,]_{u^2}^n\,[\,t+1\,]_u\, x^t \\ &= (x;u)_{2n+2}/(1-u)\Big(\sum_{t\geq 0} [\,t+1\,]_{u^2}^n - u\sum_{t\geq 0} [\,t+1\,]_{u^2}^n\,(ux)^t\Big) \\ &= (x;u)_{2n+2}/(1-u)\Big((x;u^2)_{n+1}\mathbf{E}_n(x,u^2) - u(ux;u^2)_{n+1}\mathbf{E}_n(ux,u^2)\Big) \\ &= ((xu;u^2)_{n+1}\mathbf{E}_n(x,u^2) - u(x;u^2)_{n+1}\mathbf{E}_n(ux,u^2))/(1-u), \end{aligned}$$

thus proving Theorem 11.

4.7 Interpreting q-series products with $\mathbf{s} = (2, 1, 2, 1, \ldots)$

In this section we consider the sequence $\mathbf{s} = (2, 1, 2, 1, \ldots)$. The significance of this sequence was recognized in [12] and [23] where it was shown that enumeration of these $\mathbf{s}$-lecture hall partitions is related to the q-Gauss summation and the little Göllnitz identities. We can now derive new properties from Theorem 6.

Lemma 3.

$$g(n) \triangleq \sum_{e \in \mathbf{I}_{2n}^{(2,1,2,1,\ldots,2,1)}} x^{\operatorname{asc} e}\, u^{\operatorname{amaj} e}\, q^{\operatorname{lhp} e}\, z^{|\,e\,|} = \prod_{i=1}^{n} (1 + x u^{2i} q^{3i-1} z).$$

Proof. To simplify notation in the proof, let $B_n = \mathbf{I}_{2n}^{(2,1,2,1,\ldots)}$. If $n = 1$, $B_1 = \{(0,0),\ (1,0)\}$ and $g(n) = 1 + xu^2q^2z$.

When $n > 1$, $B_n = 00B_{n-1} \cup 10B_{n-1}$. For $e \in B_{n-1}$, $00e$ and e have all statistics the same. But for $10e$, we have: $\operatorname{asc} 10e = \operatorname{asc} e + 1$; $\operatorname{amaj} 10e = \operatorname{amaj} e + 2n$; $\operatorname{lhp} 10e = \operatorname{lhp} e + 3n - 1$; and $|10e| = |e| + 1$. So, $g(n) = (1 + xu^{2n}q^{3n-1}z)g(n-1)$ and the result follows. ☐

Now use Theorem 6 with $\mathbf{s} = (2, 1, 2, 1, \ldots, 2, 1)$ and apply Lemma 3:

Theorem 12.

$$\sum_{\lambda \in \mathbf{L}_{2n}^{(2,1,\ldots,2,1)}} u^{|\lceil \lambda \rceil|}\, q^{|\lambda|}\, z^{|\,\epsilon^+(\lambda)\,|}\, x^{\lambda_{2n}} = \prod_{i=1}^{n} \frac{(1 + xu^{2i}q^{3i-1}z)}{(1 - xu^{2i}q^{3i})(1 - xu^{2i-1}q^{3i-2})}. \tag{25}$$

With four independent parameters, Theorem 12 has considerable power to interpret various finite products in terms of statistics on $(2, 1, \ldots, 2, 1)$-lecture hall partitions. For example, setting $x = u = z = 1$ in (25) gives the following from [12].

$$\sum_{\lambda \in \mathbf{L}_{2n}^{(2,1,2,1,\ldots,2,1)}} q^{|\lambda|} = \prod_{i=1}^{n} \frac{(1 + q^{3i-1})}{(1 - q^{3i})(1 - q^{3i-2})} = \frac{(-q^2; q^6)_n}{(q; q)_{3n}}.$$

Setting $u = 1/q$ and $z = -1$ in (25) gives

$$\sum_{\lambda \in \mathbf{L}_{2n}^{(2,1,2,1,\ldots,2,1)}} (-1)^{|\,\epsilon^+(\lambda)\,|}\, q^{|\lambda| - |\lceil \lambda \rceil|}\, x^{\lambda_{2n}} = \prod_{i=1}^{n} \frac{1}{1 - xq^i}.$$

Setting $x = 1/q$ and $u = z = q$,

$$\sum_{\lambda \in \mathbf{L}_{2n}^{(2,1,2,1,\ldots,2,1)}} q^{|\lambda| + |\lceil \lambda \rceil| + |\,\epsilon^+(\lambda)\,| - \lambda_{2n}} = \prod_{i=1}^{n} \frac{(1 + q^{5i-1})}{(1 - q^{5i-1})(1 - q^{5i-4})}.$$

4.8 Signed permutations and lecture hall polytopes

As another application of Theorem 5, we show that for suitable choice of $\mathbf{s}$, the $\mathbf{s}$-inversion sequences provide an alternative model for signed permutations.

It follows from (9) that the $(2,4,\ldots,2n)$-lecture hall polytope has Ehrhart polynomial $(2t+1)^n$. Thus by Theorem 5,

$$\sum_{t\geq 0}(2t+1)^n x^t \;=\; \frac{\mathbf{E}_n^{(2,4,\ldots,2n)}(x)}{(1-x)^{n+1}} \;=\; \frac{\sum_{e\in \mathbf{I}_n^{(2,4,\ldots,2n)}} e^{\operatorname{asc} e}}{(1-x)^{n+1}}. \tag{26}$$

A more familiar form for this Ehrhart series is Steingrímsson's generating function

$$\sum_{t\geq 0}(2t+1)^n x^t \;=\; \frac{\sum_{\pi\in B_n} t^{\operatorname{des}_{\mathrm{B}}(\pi)}}{(1-x)^{n+1}}.$$

where B_n is the hyperoctahedral group of signed permutations and $\operatorname{des}_{\mathrm{B}}$ is an associated descent statistic [27, 28]. This suggests a *new view of signed permutations in terms of inversion sequences and lecture hall polytopes*. In work in preparation with Thomas Pensyl, we show a correspondence between signed permutation statistics and inversion sequence statistics and demonstrate how to use our main theorem to derive results, some familiar, some new.

4.9 The sequences $(1,4,3,8,5,12,\ldots)$ and $(1,1,3,2,5,3,\ldots)$

We were originally led to study the geometry of the lecture hall polytope as an attempt to understand why, for certain special sequences $\mathbf{s}$, the generating function for the $\mathbf{s}$-lecture hall partitions has a simple product form. In this and the next two subsections we will consider such sequences from the point of Ehrhart theory and inversion statistics.

We consider two sequences that arose in [12], where it was shown that their associated lecture hall partitions had simple generating functions related to the little Göllnitz products. First, define $\mathbf{s}$ by $s_{2i}=4i,\; s_{2i-1}=2i-1$, so that $\mathbf{s}=(1,4,3,8,5,12,\ldots)$.

In new work here, we explicitly compute the Ehrhart polynomial of these $\mathbf{s}$-lecture hall polytopes. It can be checked with Theorem 3 that:

$$\mathbf{f}_n^{(\mathbf{s})}(t) \;=\; (t+1)^{\lceil n/2\rceil}(2t+1)^{\lfloor n/2\rfloor}. \tag{27}$$

Thus, by Theorem 5, with $u=q=z=1$, we have the following.

Theorem 13.

$$\sum_{t\geq 0}(t+1)^{\lceil n/2\rceil}(2t+1)^{\lfloor n/2\rfloor}x^t \;=\; \frac{\sum_{e\in \mathbf{I}_n^{(1,4,3,8,5,12,\ldots)}} x^{\operatorname{asc} e}}{(1-x)^{n+1}}.$$

If $\mathbf{s}$ is instead defined by $s_{2i}=i,\; s_{2i-1}=2i-1$, it can be checked that the Ehrhart polynomial is

$$\mathbf{f}_n^{(\mathbf{s})}(t) \;=\; (t+1)^{\lceil n/2\rceil}\left(\frac{(t+2)}{2}\right)^{\lfloor n/2\rfloor}, \tag{28}$$

so by Theorem 5, we have the following.

Theorem 14.

$$\sum_{t\geq 0}(t+1)^{\lceil n/2\rceil}\left(\frac{(t+2)}{2}\right)^{\lfloor n/2\rfloor}x^t \;=\; \frac{\sum_{e\in\mathbf{I}_n^{(1,1,3,2,5,3,\ldots)}} x^{\operatorname{asc} e}}{(1-x)^{n+1}}.$$

The Ehrhart polynomials (27) and (28) for these sequences seem to be a blend of (9) and (26). But in contrast to those cases, we did not see a natural u-analog of (27) and (28) in our computations.

4.10 The sequences $\mathbf{s} = (1, k + 1, 2k + 1, \ldots)$

For integer $k \geq 1$, the sequences $\mathbf{s} = (1, k+1, 2k+1, \ldots)$ give rise to a special generalization of the familiar Eulerian polynomials. Note that when $k = 1$, $\mathbf{s}$ is the sequence of positive integers and by Lemma 1, $\mathbf{E}_n^{(\mathbf{s})} = \mathbf{E}_n(x)$. We will use Corollary 2(a) to show that the new lecture hall statistic has a nice distribution for these $\mathbf{s}$-inverions sequences.

We appeal to a result of Bousquet-Mèlou and Eriksson from [3], which shows that there is a generating function for these $\mathbf{s}$-lecture hall partitions:

Theorem 15. (Special case of Proposition 4 in [3])

$$\sum_{\lambda\in\mathbf{L}_n^{(1,k+1,2k+1,\ldots,(n-1)k+1)}} q^{|\lambda|} \;=\; \prod_{i=1}^{n}\frac{1}{1-q^{2i-1}}\prod_{i=\lceil n/2\rceil}^{n}\frac{1-q^{(2i-1)((n-i)k+1)}}{1-q^{(n-i+1)((2i-3)k+2)}}. \qquad (29)$$

Setting $k = 1$ gives (1). Combining with Corollary 2(a):

Corollary 11.

$$\sum_{e\in\mathbf{I}_n^{(1,k+1,2k+1,\ldots,(n-1)k+1)}} q^{\operatorname{lhp} e} \;=\; \prod_{i=1}^{n}[\,(n-i+1)k+1\,]_{q^{2i-1}}\,.$$

So, for example, the distribution of the lhp statistic over the $(1, 5, 9, 13)$-inversion sequences is

$$[\,1\,]_{q^7}\,[\,5\,]_{q^5}\,[\,9\,]_{q^3}\,[\,13\,]_{q}\,.$$

In recent work it has been shown that the sequences $\mathbf{s} = (1, k + 1, 2k + 1, \ldots)$ give rise to an analytic generalization of the Eulerian polynomials and a new combinatorial interpretation [24]. The key was the discovery and proof in [24] of an explicit formula for the Ehrhart polynomial of the $\mathbf{s}$-lecture hall polytope.

In contrast, we have been unable to do an analogous computation for an intriguing generalization of the Eulerian polynomials described in the next section.

4.11 The ℓ-Eulerian polynomials

Fix integer $\ell \geq 2$ and define the *ℓ-sequence* $\{a_n^{(\ell)}\}$ by $a_n^{(\ell)} = \ell a_{n-1}^{(\ell)} - a_{n-2}^{(\ell)}$, with initial conditions $a_0^{(\ell)} = 0$ and $a_2^{(\ell)} = 1$. For example, $\{a_n^{(2)}\} = (0, 1, 2, 3, 4, \ldots)$ and $\{a_n^{(3)}\} = (0, 1, 3, 8, 21, 55 \ldots)$. Bousquet-Mèlou and Eriksson proved a generalized lecture hall theorem in [3], which includes the following special case:

Theorem 16. The ℓ-Lecture Hall Theorem [3]: *For* $\mathbf{s} = (a_1^{(\ell)}, a_2^{(\ell)}, \ldots, a_n^{(\ell)})$,

$$L_n^{(\mathbf{s})}(q) \ \triangleq \ \sum_{\lambda \in \mathbf{L}_n^{(\mathbf{s})}} q^{|\lambda|} \ = \ \prod_{i=1}^{n} \frac{1}{1 - q^{a_i^{(\ell)} + a_{i-1}^{(\ell)}}}.$$

Setting $\ell = 2$ gives (1). (Bousquet-Mèlou and Eriksson found that in the limit as $n \to \infty$, Theorem 16 gives a surprising generalization of Euler's partition theorem [3, 25].)

Combining Theorem 16 with the first identity in Corollary 2, shows that the lhp statistic has a nice distribution when $\mathbf{s}$ is an ℓ-sequence, even when $\ell > 2$. (Compare to the $\ell = 2$ case in Corollary 5.)

Corollary 12. *For* $\mathbf{s} = \{a_n^{(\ell)}\}_{n \geq 1}$,

$$\sum_{e \in \mathbf{I}_n^{(\mathbf{s})}} q^{\mathrm{lhp}\, e} \ = \ \prod_{i=1}^{n} \left[a_i^{(\ell)} \right]_{q^{a_i^{(\ell)} + a_{i-1}^{(\ell)}}} .$$

So, for example, the distribution of the lhp statistic over the $(1, 3, 8, 21)$-inversion sequences is

$$[1]_{q^{27}} \, [3]_{q^{11}} \, [8]_{q^{4}} \, [21]_{q} \, .$$

When $\mathbf{s} = \{a_n^{(\ell)}\}_{n \geq 1}$ we can use Theorem 3 together with polynomial interpolation to compute the Ehrhart polynomial for the $\mathbf{s}$-lecture hall polytope for fixed n and ℓ. For example, when $\ell = 3$, for $n = 6$, the Ehrhart polynomial is

$$\mathbf{f}_n^{(\mathbf{s})}(t) \ = \ 5544t^6 + 23100t^5 + 34188t^4 + 20680t^3 + 7659t^2/2 - 435t/2 + 1.$$

Recall from (9) that when $\ell = 2$ the Ehrhart polynomial is $\mathbf{f}_n^{(\mathbf{s})}(t) = (t+1)^n$. Unfortunately, we have not found a closed form when $\ell > 2$. If we define the "ℓ-Eulerian polynomials" by

$$E_n^{(\ell)}(x) \ = \ \sum_{e \in \mathbf{I}_n^{\mathbf{s}}} x^{\mathrm{asc}\, e},$$

then $E_n^{(\ell)}(x)$ is a generalization of the familiar Eulerian polynomials $\mathbf{E}_n(x) = E_n^{(2)}(x)$. Shown below is the triangle of ℓ-Eulerian numbers for $\ell = 3$.

1
1 2
1 16 7
1 150 309 44
1 2504 14922 9764 529
1 87118 1211494 2096432 583697 12938

5 Connections to previous work on lecture hall partitions

Bousquet-Mèlou and Eriksson recognized from the beginning [2] that the generating function for **s**-lecture hall partitions would have the form

$$\sum_{\lambda \in L_n^{(\mathbf{s})}} q^{|\lambda|} = \frac{p_n(q)}{\prod_{i=1}^{n}(1 - q^{s_{i+1}+\cdots+s_n})},$$

where $p_n(q)$ was a polynomial counting what they referred to as *reduced lecture hall partitions*. Their focus was on finding infinite families of sequences, **s**, for which $p_n(q)$ could be computed in closed form. One of our main contributions in the current paper is to show that the numerator polynomial can be defined in terms of inversion sequences and thereby, via the ascent statistic and Ehrhart theory, be interpreted in the geometry of the corresponding polytope.

An "explicit" form for $\sum_{\lambda \in L_n^{(\mathbf{s})}} q^{|\lambda|}$, was proved in [11]. Although we referred to it as the "horrible generating function", we used it for many years as a computational tool before making the connection that we were computing statistics on inversion sequences. A multivariate form was also computed in [11], and the methods in this paper could be adapted to do the same.

Bousquet-Mèlou and Eriksson were the ones who discovered the importance of $\lceil \lambda \rceil$ and the statistic $|\lceil \lambda \rceil|$ tracked by u [4]. Following up on a suggestion of Vic Reiner, they were looking for a "lecture hall partition" statistic to correspond to a statistic u appearing in a refinement of Bott's formula. Bousquet-Mèlou and Eriksson's refined lecture hall theorem included a third variable, v, which kept track of the number of odd parts of $\lceil \lambda \rceil$. This could also be translated in terms of inversion sequences for $\mathbf{s} = (1, 2, \ldots, n)$ and perhaps generalized.

We first interpreted the generating function for refined lecture hall partitions and anti-lecture hall compositions in terms of statistics on permutations (quadratic permutation statistics corresponding to $\sum_{j=i}^{n} j$ and to $\sum_{j=i}^{n}(n-j)$) in [5]. Inversion sequences were introduced in the current paper as a way of generalizing those results to arbitrary sequences **s**. In retrospect, inversion sequences give a more natural interpretation.

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