On the Existence of Hamiltonian Paths in the Cover Graph of $M(n)$

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Abstract

The poset $M(n)$ has as its elements the $n$-tuples of integers $a = (a_1, a_2, \ldots, a_n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ for some $j$, $0 \leq j \leq n$. The order relation is defined by $a \leq b$ iff $a_i \leq b_i$ for $1 \leq i \leq n$. We show that the cover graph of $M(n)$ has a Hamiltonian path if and only if $\binom{n+1}{2}$ is odd and $n \neq 5$.

Keywords: Hamiltonian path, Gray code, cover graph, augmentation poset

1 Introduction

Introduced by Stanley [9], the poset $M(n)$ has as its elements the $n$-tuples of integers $a = (a_1, a_2, \ldots, a_n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ for some $j \geq 0$. The order relation is defined by $a \leq b$ iff $a_i \leq b_i$ for $1 \leq i \leq n$. A discussion of order properties of posets appears in [10]; a nice description of $M(n)$ in particular appears in [6]. Another description of $M(n)$ is as the lattice of order ideals in the product of chains of sizes 2 and $n-1$. The poset $M(n)$ is both ranked and order-symmetric. It has the Sperner property, and it is rank-unimodal [9, 6]. It is unknown whether $M(n)$ has a symmetric chain decomposition.

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In contrast to the order-theoretic properties of $M(n)$, in this paper we consider graph-theoretic properties of its Hasse diagram. When viewed as an undirected graph, the Hasse diagram is called the cover graph of the poset. Vertices are adjacent in the cover graph if they are related elements of the poset with no other elements between them. As illustrated in Figure 1, the elements of $M(n)$ can be viewed as the subsets of the set $[n] = \{1, 2, \ldots, n\}$. In this phrasing, two vertices are adjacent in the cover graph if one is obtained from the other by adding the element 1 or by augmenting one element by one. We prove that the cover graph of $M(n)$ has a Hamiltonian path if and only if $\binom{n+1}{2}$ is odd and $n \neq 5$. We set up the main result in Section 2 and describe the construction in Section 3.

For comparison and motivation, we mention a more familiar partial order on the same set: the inclusion relation. This defines a poset on the subsets of $[n]$, called the Boolean lattice $B(n)$. Its cover graph is the $n$-cube. The $n$-cube is Hamiltonian and has Hamiltonian paths satisfying a wide variety of constraints (e.g. $[2, 3, 4, 5, 7]$). It is easy to obtain a Hamiltonian path in $B(n)$ by induction. The cover graph of $M(n)$ has a somewhat more complicated structure than that of $B(n)$ and fewer edges: $(n+1)2^{n-2}$ instead of $n2^{n-1}$.
2 Necessary Conditions

Let $A_n$ be the cover graph of $M(n)$. For compactness, we will represent the vertices of $A_n$ as subsets of $[n]$. We write element $a \in M(n)$ satisfying $0 = a_1 = \cdots = a_j < a_{j+1} < \cdots a_n \leq n$ as the vertex $\{a_{j+1}, \ldots, a_n\}$ of $A_n$, as in Figure 1.

For sets $X, Y$, let $X \oplus Y = (X \cup Y) \setminus (X \cap Y)$. In $A_n$ vertices $X$ and $Y$ are adjacent if and only if

(i) $|X| = |Y|$ and $X \oplus Y = \{i, i+1\}$ for some $i$ or

(ii) $X \oplus Y = \{1\}$.

For a finite set $S$, let $\sigma(S)$ denote the sum of the elements of $S$. Note that if $X$ and $Y$ are adjacent in $A_n$ then $\sigma(X)$ and $\sigma(Y)$ differ by 1. Thus $A_n$ is bipartite with bipartition $(E_n, O_n)$, where

$$E_n = \{X \in 2^{[n]} : \sigma(X) \text{ is even}\}$$

$$O_n = \{X \in 2^{[n]} : \sigma(X) \text{ is odd}\}$$

**Proposition 1** For $n \geq 1$, if $\binom{n+1}{2}$ is even, then $A_n$ does not have a Hamiltonian path.

**Proof.** Always $|E_n| = |O_n|$, so a Hamiltonian path in $A_n$ must have one endpoint in $E_n$ and the other in $O_n$. On the other hand, $A_n$ has two vertices of degree 1, namely $\emptyset$ and $[n]$, which therefore must be the endpoints of any Hamiltonian path. If $\sigma([n]) = \binom{n+1}{2}$ is even, then the vertices that must be the endpoints both lie in $E_n$. □

When $n = 5$, $\binom{n+1}{2}$ is odd, but $A_5$ does not have a Hamiltonian path. To see this, note in Figure 2 that vertices of degree 2 in $A_n$ successively force a Hamiltonian path in $A_5$ to use the edges on the disjoint paths (shown in boldface on the figure):

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{3\}, \{4\}, \{5\}, \{1,5\}, \{1,4\}, \{2,4\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}$$

and

$$\{1,2,3,4,5\}, \{2,3,4,5\}, \{1,3,4,5\}, \{3,4,5\}, \{2,4,5\}, \{1,2,4,5\}, \{1,2,3,5\},$$

$$\{1,2,3,4\}, \{2,3,4\}, \{2,3,5\}, \{1,3,5\}, \{1,4,5\}, \{4,5\}, \{3,5\}.$$  

However, the subgraph of $A_n$ induced by $\{1,2,4\}, \{3,5\}, \{2,5\}, \{3,4\}, \{1,2,5\}$, and $\{1,3,4\}$ is a 6-cycle in which $\{1,2,4\}$ and $\{3,5\}$ are diametrically opposite. Hence the path cannot be completed.

In the remainder of this paper, we prove the following theorem.

**Theorem 1** For $n \geq 1$, if $\binom{n+1}{2}$ is odd and $n \neq 5$, then $A_n$ has a Hamiltonian path.

This is clear for $n \leq 2$. For $n > 5$, the result will follow from Lemma 5 in the next section.
Figure 2: $A_5$, the cover graph of $M(5)$. 
3 Construction of Paths

For \( n \geq 3 \), let \( G_n \) be the subgraph of \( A_n \) induced by the vertices \( V(G_n) = V(A_n) \setminus \{\emptyset, \{1\}, \{2, 3, \ldots, n\}, [n]\} \). Note that \( A_n \) has a Hamiltonian path if and only if \( G_n \) has a Hamiltonian path from \( \{2\} \) to \( \{1, 3, \ldots, n\} \). We will prove via a sequence of lemmas that such a path exists in \( G_n \) whenever \( \binom{n+1}{2} \) is odd and \( n \neq 5 \). Our strategy is to build up paths and cycles in subgraphs of \( G_n \) that we will combine to form the desired Hamiltonian path.

For \( n \geq 3 \), let \( e_n \) and \( a_n \) denote the edges in \( G_n \):

\[
e_n = \{1, 2, \ldots, n-1\}\{2, \ldots, n-1\}
\]

\[
a_n = \{n\}\{1, n\}.
\]

(We use \( uv \) to denote the edge joining adjacent vertices \( u \) and \( v \).)

**Lemma 1** For \( n \geq 3 \), every Hamiltonian cycle in \( G_n \) contains the following edges:

\[
\begin{align*}
(a) & \quad \{2\}\{3\} \\
(b) & \quad \{2\}\{1, 2\} \\
(c) & \quad \{1, 2\}\{1, 3\} \\
(d) & \quad \{n\}\{1, n\} = a_n \\
(e) & \quad [n-1]\{2, 3, \ldots, n-1\} = e_n.
\end{align*}
\]

Furthermore, for each edge \( uv \) in (a)-(e) every Hamiltonian path in \( G_n \) that does not start or end at \( u \) or \( v \) contains the edge \( uv \).

**Proof.** This holds because each of the vertices \( \{1, 2, \ldots, n-1\}, \{n\}, \{2\}, \) and \( \{1, 2\} \) has degree 2 in \( G_n \). □

When \( H \) is a subgraph of \( G_{n-1} \), let \( nH \) denote the subgraph of \( G_n \) obtained from the graph \( H \) by replacing each vertex \( X \) of \( H \) by the vertex \( X \cup \{n\} \). For an edge \( e = XY \), let \( V(e) \) denote the set \( \{X, Y\} \). Then the vertex set of \( G_n \) is the disjoint union

\[
V(G_n) = V(G_{n-1}) \cup V(nG_{n-1}) \cup V(e_n) \cup V(a_n).
\]

(1)

Let \( P \) be a path in \( G_n \) containing edge \( uv \), and let \( e = wz \) be an edge of \( G_n \) with neither \( w \) nor \( z \) on \( P \). If \( uw \) and \( vz \) are also edges of \( G_n \), then by “pulling \( e \) into \( P \)” we will mean replacing the edge \( uv \) of \( P \) by the path \( u, w, z, v \) to obtain a new path \( P' \) with \( V(P') = V(P) \cup V(e) \).
Figure 3: Construction of $C_n$ in Lemma 2 when $C_{n-1}$ contains $\{2, n-1\}\{3, n-1\}$.

Figure 4: Construction of $C_n$ in Lemma 2 when $C_{n-1}$ contains $\{1, 2, n-1\}\{1, 3, n-1\}$. 
Note that if $P$ and $Q$ are Hamiltonian paths in $G_{n-1}$, then by Lemma 1 we can pull $a_n$ into $P$ as long as $P$ does not start or end with \{n \mid 0 \}$ or \{1 \mid n \} and we can pull $e_n$ into $nQ$ as long as $Q$ does not start or end with \[n-2 \mid 0 \} \} or \{2 \mid 3 \ldots, n-2 \}.

**Lemma 2** For $n \geq 3$, $G_n$ has a Hamiltonian cycle $C_n$. For $n \geq 4$, there is such a cycle containing the edge $\{2 \mid n \} \{3 \mid n \}$ and such a cycle containing the edge $\{1 \mid 2 \mid n \} \{1 \mid 3 \mid n \}$.

**Proof.** For $n = 3$, $G_3$ itself is the cycle $C_3$ (of length 4). For $n = 4$, the cycle $C_4$ below contains both special edges.

$$\{2 \mid , \{1 \mid 2 \}, \{1 \mid 3 \}, \{2 \mid 3 \}, \{1 \mid 2 \mid 4 \}, \{1 \mid 3 \mid 4 \}, \{3 \mid 4 \}, \{2 \mid 4 \}, \{1 \mid 4 \}, \{4 \} \}, \{3 \}$$

Assume that $n > 4$, and let $C_{n-1}$ be a Hamiltonian cycle in $G_{n-1}$ satisfying one of the claimed conditions of the theorem. By (1), we can form $C_n$ from $C_{n-1}$ and $nC_{n-1}$ as follows. Pull edge $e_n$ into $nC_{n-1}$ and edge $a_n$ into $C_{n-1}$ and then link the cycles together as described below.

If $C_{n-1}$ contains edge $\{2 \mid n \} \{3 \mid n \}$, then delete this edge from $C_{n-1}$ and delete edge $\{2 \mid n \} \{3 \mid n \}$ from $nC_{n-1}$ (guaranteed to exist by Lemma 1). Add edges $\{2 \mid n \} \{2 \mid n \}$ and $\{3 \mid n \} \{3 \mid n \}$. (See Figure 3. In this and other figures vertices need not occur on cycles and paths in the order shown.) The resulting cycle contains edge $\{1 \mid 2 \mid n \} \{1 \mid 3 \mid n \}$, since Lemma 1 implies that $C_{n-1}$ contains edge $\{1 \mid 2 \} \{1 \mid 3 \}.$

In the other case, $C_{n-1}$ contains the edge $\{1 \mid 2 \mid n \} \{1 \mid 3 \mid n \}$. Delete this edge from $C_{n-1}$ and delete edge $\{1 \mid 2 \mid n \} \{1 \mid 3 \mid n \}$ from $nC_{n-1}$ (guaranteed to exist by Lemma 1.) Add edges $\{1 \mid 2 \mid n \} \{1 \mid 2 \mid n \}$ and $\{1 \mid 3 \mid n \} \{1 \mid 3 \mid n \}$. (See Figure 4.) The resulting cycle contains edge $\{2 \mid n \} \{3 \mid n \}$, since Lemma 1 implies that $C_{n-1}$ contains edge $\{2 \} \{3 \}. \square$

**Lemma 3** For $n \geq 5$, $G_n$ has a Hamiltonian path $Q_n$ from $\{2 \}$ to $\{2 \mid n \}$ if $n$ is odd and from $\{2 \}$ to $\{1 \mid 2 \mid n \}$ if $n$ is even.

**Proof.** For $n = 5$, a path $Q_5$ satisfying the requirements is shown in Figure 5 (compare with Figure 2). Assume $n > 5$. If $n$ is odd, then the induction hypothesis implies that $G_{n-1}$ has a Hamiltonian path $Q_{n-1}$ from $\{2 \}$ to $\{1 \mid 2 \mid n \}$. By Lemma 2, $nG_{n-1}$ has a Hamiltonian cycle $nC_{n-1}$. By (1) we can form $Q_n$ from $Q_{n-1}$ and $nC_{n-1}$ by pulling edge $e_n$ into $nC_{n-1}$ and edge $a_n$ into $Q_{n-1}$ and then linking the resulting path and cycle as follows. By Lemma 1, $nC_{n-1}$ contains the edge $\{2 \mid n \} \{1 \mid 2 \}$ and $\{1 \mid 2 \mid n \} \{1 \mid 2 \mid n \}$. Delete this edge and add the edge from the endpoint $\{1 \mid 2 \mid n \}$ of $Q_{n-1}$ to the vertex $\{1 \mid 2 \mid n \}$ of $nC_{n-1}$. The result is a Hamiltonian path $Q_n$ in $G_n$ from $\{2 \}$ to $\{1 \mid 2 \mid n \}$ (Figure 6).

If $n$ is even, then the induction hypothesis implies that $G_{n-1}$ has a Hamiltonian path $Q_{n-1}$ from $\{2 \}$ to $\{2 \mid n \}$. Construct $Q_n$ as in the odd case, except in the final step, add the edge from the endpoint $\{2 \mid n \} \{Q_{n-1}$ to the vertex $\{2 \mid n \}$ of $nC_{n-1}$, giving a Hamiltonian path $Q_n$ in $G_n$ from $\{2 \}$ to $\{2 \mid n \}$ (Figure 6). \square
$Q_5 =$
\{2\}, \{1, 2\}, \{1, 3\}, \{3\}, \{4\}, \{5\}, \{1, 5\},
\{1, 4\}, \{2, 4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{3, 4\},
\{3, 5\}, \{4, 5\}, \{1, 4, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\},
\{1, 2, 3, 5\}, \{1, 2, 3, 4\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 3, 5\}, \{1, 2, 5\}, \{2, 5\}

Figure 5: The Hamiltonian path $Q_5$ in $G_5$ (read across) in Lemma 3.

Figure 6: Construction of $Q_n$ in Lemma 3.
For the basis case $n = 5$, a path $Q_5^*$ satisfying the requirements is shown in Figure 7 (compare with Figure 2). For $n > 5$, $Q_n^*$ is constructed from $Q_{n-1}^*$ and $nC_{n-1}$ exactly as $Q_n$ is constructed from $Q_{n-1}$ and $nC_{n-1}$ in the proof of Lemma 3. (See Figure 8). □

**Lemma 5** For $n \geq 6$, $G_n$ has a Hamiltonian path $P_n$ from $\{2\}$ to $\{1, 3, 4, \ldots, n\}$ if $\binom{n+1}{2}$ is odd and from $\{2\}$ to $\{3, 4, \ldots, n\}$ if $\binom{n+1}{2}$ is even. In addition, when $n \geq 6$ and $\binom{n+1}{2}$ is odd, $G_n$ has a Hamiltonian path $R_n$ from $\{1, 2\}$ to $\{3, 4, \ldots, n\}$.

**Proof.** For the basis case $n = 6$, paths $P_6$ and $R_6$ satisfying the requirements are given in...
$P_6 =$

(2), (1, 2), (1, 3), (3), (4), (1, 4), (2, 4),
(2, 3), (2, 3), (1, 2, 4), (1, 2, 5), (2, 5), (1, 5), (5),
(6), (1, 6), (2, 6), (1, 2, 6), (1, 3, 6), (3, 6), (4, 6),
(5, 6), (1, 5, 6), (2, 5, 6), (1, 2, 4, 6), (2, 3, 6), (2, 3, 4), (4, 6),
(2, 4, 6), (1, 4, 6), (1, 4, 5), (4, 5), (5), (3, 4), (3, 4), (1, 3, 4),
(1, 3, 5), (2, 3, 5), (2, 3, 4), (1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 3), (2, 4, 5),
(3, 4, 5), (1, 3, 4, 5), (2, 3, 4, 5), (1, 2, 3, 4, 5), (1, 2, 3, 4, 6), (2, 3, 4, 6), (2, 3, 4, 6), (1, 3, 4, 6),
(3, 4, 6), (3, 5, 6), (4, 5, 6), (1, 4, 5, 6), (1, 3, 5, 6), (2, 3, 5, 6), (1, 2, 3, 5, 6),
(1, 2, 4, 5, 6), (2, 4, 5, 6), (3, 4, 5, 6), (1, 3, 4, 5, 6), (1, 3, 4, 5, 6), (1, 2, 4, 5, 6),

$R_6 =$

(1, 2), (2), (3), (4), (5), (6), (1, 6),
(2, 6), (1, 2, 6), (1, 2, 5), (2, 5), (1, 5), (1, 4), (1, 3), (1, 3),
(2, 3), (1, 2, 3), (1, 2, 4), (2, 4), (3, 4), (1, 3, 4), (2, 3, 4),
(1, 2, 3, 4), (1, 2, 3, 5), (1, 2, 4, 6), (1, 2, 5, 6), (2, 5, 6), (1, 5, 6),
(5, 6), (4, 6), (3, 6), (1, 3, 6), (1, 4, 6), (2, 4, 6), (2, 3, 6),
(2, 3, 5), (1, 3, 5), (3, 5), (4, 5), (1, 4, 5), (2, 4, 5), (1, 2, 4, 5),
(1, 3, 4, 5), (3, 4, 5), (3, 4, 6), (1, 3, 4, 6), (1, 3, 5, 6), (3, 5, 6), (4, 5, 6),
(1, 4, 5, 6), (2, 4, 5, 6), (2, 3, 5, 6), (2, 3, 4, 6), (2, 4, 5), (1, 2, 3, 4, 5), (1, 2, 3, 4),
(1, 2, 3, 5, 6), (1, 2, 4, 5, 6), (1, 3, 4, 5, 6), (3, 4, 5, 6)

Figure 9: The Hamiltonian paths $P_6$ and $R_6$ (read across) in Lemma 5.

$Q_{\nicefrac{n}{2}}$ $nP_{\nicefrac{n}{2}}$

Figure 10: Construction of $P_n$ in Lemma 5 when $\binom{n+1}{2}$ is even and $n$ is (a) odd, (b) even.
Figure 9. Assume \( n > 6 \). We consider four cases, according to whether \( \binom{n+1}{2} \) and \( n \) are, independently, odd or even.

Case 1. If \( \binom{n+1}{2} \) is even and \( n \) is odd, then \( \binom{n-1+1}{2} \) is odd. By Lemma 3, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1} \) from \( \{2\} \) to \( \{1, 2, n - 1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nR_{n-1} \) from \( \{1, 2, n\} \) to \( \{3, 4, \ldots, n - 1, n\} \). Construct \( P_n \) by pulling edge \( e_n \) into \( nR_{n-1} \) and edge \( a_n \) into \( Q_{n-1} \). Now link \( Q_{n-1} \) to \( nR_{n-1} \) by adding edge \( \{1, 2, n - 1\}\{1, 2, n\} \). See Figure 10(a).

Case 2. If \( \binom{n+1}{2} \) is even and \( n \) is even, then by Lemma 3, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1} \) from \( \{2\} \) to \( \{2, n - 1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nP_{n-1} \) from \( \{2, n\} \) to \( \{3, 4, \ldots, n - 1, n\} \). Construct Hamiltonian path \( P_n \) in \( G_n \) by pulling edge \( e_n \) into \( nP_{n-1} \) and edge \( a_n \) into \( Q_{n-1} \) and then adding edge \( \{2, n - 1\}\{2, n\} \). See Figure 10(b).

Case 3. If \( \binom{n+1}{2} \) is odd and \( n \) is odd, then we must construct both \( P_n \) and \( R_n \). We first construct \( P_n \). By Lemma 3, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1} \) from \( \{2\} \) to \( \{1, 2, n - 1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nP_{n-1} \) from \( \{2, n\} \) to \( \{3, 4, \ldots, n\} \). We take the complement in \( n - 1 \) of every set on the path \( P_{n-1} \) to get a path \( P_{n-1}^c \) from \( \{1, 3, 4, \ldots, n-1\} \) to \( \{1, 2\} \). Now \( nP_{n-1}^c \) is a Hamiltonian path in \( G_{n-1} \) from \( \{1, 3, 4, \ldots, n-1, n\} \) to \( \{1, 2, n\} \). Note that \( nP_{n-1}^c \) must have edge \( \{1, 2, \ldots, n - 2, n\}\{2, 3, \ldots, n - 2, n\} \) by Lemma 1. To construct \( P_n \), pull edge \( a_n \) into \( Q_{n-1} \) and edge \( e_n \) into \( nP_{n-1} \), and then join the two paths with the edge from \( \{1, 2, n - 1\} \) to \( \{1, 2, n\} \) (Figure 11).

![Figure 11: Construction of Pn and Rn in Lemma 5 when \( \binom{n+1}{2} \) is odd and n is odd.](image-url)
Next we construct \( R_n \). By Lemma 4, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1}^* \) from \( \{1, 2\} \) to \( \{2, n-1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nP_{n-1} \) from \( \{2, n\} \) to \( \{3, 4, \ldots, n-1, n\} \). Pull edge \( e_n \) into \( nP_{n-1} \) and edge \( a_n \) into \( Q_{n-1}^* \), and then join the paths by adding the edge \( \{2, n-1\}\{2, n\} \) (Figure 11).

Case 4. If \( \binom{n+1}{2} \) is odd and \( n \) is even, then we again construct \( P_n \) and then \( R_n \). By Lemma 3, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1}^* \) from \( \{2\} \) to \( \{2, n-1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nP_{n-1} \) from \( \{2, n\} \) to \( \{1, 3, 4, \ldots, n-1, n\} \). Pull edge \( e_n \) into \( nP_{n-1} \) and edge \( a_n \) into \( Q_{n-1} \), and then join the paths by adding the edge \( \{2, n-1\}\{2, n\} \) to form \( P_n \) (Figure 12).

By Lemma 4, \( G_{n-1} \) has a Hamiltonian path \( Q_{n-1}^* \) from \( \{1, 2\} \) to \( \{1, 2, n-1\} \). By the induction hypothesis, \( nG_{n-1} \) has a Hamiltonian path \( nR_{n-1} \) from \( \{1, 2, n\} \) to \( \{3, 4, \ldots, n-1, n\} \). Pull edge \( e_n \) into \( nR_{n-1} \) and edge \( a_n \) into \( Q_{n-1}^* \), and then join the paths by adding the edge \( \{1, 2, n-1\}\{1, 2, n\} \) to form \( R_n \) (Figure 12). \( \square \)

Theorem 1 now follows: when \( \binom{n+1}{2} \) is odd, inserting \( \emptyset, \{1\} \) at the beginning and appending \( \{2, 3, \ldots, n\}, \{1, 2, \ldots, n\} \) to the end of \( Q_n \) completes a Hamiltonian path in the full graph \( A_n \).

The Hamiltonian path software described in [8] was most useful in allowing us initially to test the claim of Theorem 1 and then to find candidate Hamiltonian paths in \( G_n \) for the inductive construction.
References


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