MA747 Midterm (Take Home) Solns

1. $E|S_T| \le E\sum_{i=1}^{T} |Y_i|$

(T finite) $= \lim_{n\to\infty} E\sum_{i=1}^{T\wedge n} |Y_i|$

$= \lim_{n\to\infty} E\sum_{i=1}^{n} |Y_i| \, 1_{\{i\le T\}}$

(MCT) $= E\sum_{i=1}^{\infty} |Y_i| \, 1_{\{i\le T\}}$

$= \sum_{i=1}^{\infty} E|Y_i| \, 1_{\{i\le T\}}$ (A)

but $Y_i \in \mathcal{F}_i$ and $\{i\le T\} = \{T\le i-1\}^c \in \mathcal{F}_{i-1}$.
But clearly $Y_i \in \sigma(Y_k ; k\ge i)$ so Y_i is independent of $\{i\le T\}$.

Thus (A) $= \sum_{i=1}^{\infty} E|Y_i| P(i\le T)$

(iid), def. expectation for nonneg. r.v. $= E|Y_1|\, ET < \infty$, since both expectations are given to be $<\infty$.

2. (a) $\Rightarrow$ (b). By Mart. Converg. Thm $Z_n \to Z$ a.s. and $E|Z| < \infty$. Using the hint $|Z_n| = M_n + A_n$, $X_n = M_n + E[A_\infty | \mathcal{F}_n]$. Since $Z_n \to Z$ a.s. & $E|Z| < \infty$, it follows $A_\infty \in L_1$ so for $Y_n \triangleq E[A_\infty | \mathcal{F}_n]$ we have (Y_n) is a martingale. Hence X_n is a martingale. (Note Y_n & $M_n \in L_1$ so $X_n \in L_1$)

Note (A_n) is increasing so
$E[A_n | \mathcal{F}_n] \le E[A_\infty | \mathcal{F}_n]$ or $A_n \le Y_n$

Also
$$Z_n \le |Z_n| \le M_n + Y_n = X_n.$$

(b) $\Rightarrow$ (c). Immediate: X_n in (b) ≥ 0 & $Z_n \le |Z_n|$, so $X_n \ge Z_n$.

(c) $\Rightarrow$ (d). Let $Y_n \triangleq X_n - Z_n$. From (c), we note martingale $(Y_n) \ge 0$. Then, trivially $Z_n = X_n - Y_n$. Note we could add, for example a constant c to each X_n & Y_n & still have $Z_n = X_n - Y_n$.

(d) $\Rightarrow$ (a). Note $\sup_n E|Z_n| \le \sup_n E|X_n|$

$+ \sup_n E(Y_n)$,

but (X_n) & (Y_n) are martingales so $E|Y_n| < \infty$ & $E|X_n| < \infty$, $\forall n$. It follows

$$\sup_n E|Z_n| < \infty.$$

3\. (a) Note $1 < \frac{|X_n|}{M}$ on $\{|X_n| > M\}$,

so $\sup_n \int_{\{|X_n| > M\}} |X_n| \, dP \le \int |X_n| \left[\frac{|X_n|}{M}\right]^{\varepsilon} dP$

(using K in problem) $\longrightarrow$ $\le K M^{-\varepsilon} \to 0$ as $M \to \infty$.

(b) By Theorem A in class, X_n converges in L_1 and a.s.