Midterm Solutions

1. (X_n) mtgle so $E|X_n| < \infty$ so $E(e^{tX_n}) < \infty$. e^{tx} is convex, any $t \in \mathbb{R}$. By Jensen + mtgle: $E(e^{tX_{n+1}} | \mathcal{F}_n) \geq e^{t(E[X_{n+1}|\mathcal{F}_n])} = e^{tX_n}$ so submtgle

2. let $M_n = \sum_{i=1}^{n} Y_n - E(Y_n | \mathcal{F}_{n-1})$, $n \geq 1$. (The conditional expectation is well defined since $Y_n \in L_1$. X_n is integrable since the Y_n + $E[Y_n|\mathcal{F}_n]$ are integrable. For $n \geq 1$

$E[M_{n+1} | \mathcal{F}_n] = \sum_{j=1}^{n+1} E[Y_j - E(Y_j | \mathcal{F}_{j-1}) | \mathcal{F}_n] =$
$\sum_{j=1}^{n} Y_j - E(Y_j|\mathcal{F}_{j-1}) + (E[Y_{n+1}|\mathcal{F}_n] - E[Y_{n+1}|\mathcal{F}_n]) = M_n$.

3. Note $X_n \geq 0$ and is a supermartingale. so

(a) $E[X_{n+1} | \mathcal{F}_n] \leq X_n$. In particular, for $k \geq 2$
$E[X_{n+k} | \mathcal{F}_n] = E[E[X_{n+k} | \mathcal{F}_{n+k-1}] | \mathcal{F}_n]$
$\leq E[X_{n+k-1} | \mathcal{F}_n]$
using monotonicity property of expectation. By the above 2 results, taking $m = n+1$ or $n+k$, $k \geq 2$, and using induction in the latter case and setting $n = 0$ gives us the result.

(b) This follows immediately from the martingale convergence thm (in particular Durrett's Corollary for supermartingales).

4. $P(T < \infty) = 1$ implies $X_{T \wedge n} \to X_T$ as $n \to \infty$. Also, $|X_T| \leq K < \infty$ so $E|X_T| < \infty$. Thus,

$E|X_T - X_{T\wedge n}| \leq 2K P(T > n) \to 0$ (A)

as $n \to \infty$. Since (X_n) is a martingale,

$(X_{T\wedge n})$ is a martingale and so

$$E X_{T\wedge n} = E X_0 .$$

Using the result in (A), we have $E X_T = E X_0$.

5. Note S_n is a martingale:

$$E[S_{n+1}|\mathcal{F}_n] = E[\beta_{n+1} X_{n+1}|\mathcal{F}_n] + S_n$$
$$= \beta_{n+1}\underbrace{E[X_{n+1}|\mathcal{F}_n]}_{0} + S_n = S_n$$

Hence,
$|S_n|$ is a submartingale.

By Doob's inequality

$$P\left(\max_{1\le k\le m} |S_k| > M\right) \le \frac{E|S_m|}{M}$$